CONCEPTUAL FRAMEWORK FOR IMPROVING THE MATHEMATICAL TRAINING OF YOUNG PEOPLE

MONOGRAPH

BUDAPEST

2016

www.seanewdim.com

CONCEPTUAL FRAMEWORK

FOR IMPROVING

THE MATHEMATICAL TRAINING

OF YOUNG PEOPLE

Monograph

Edited by

prof. N. Tarasenkova

BUDAPEST

2016

Tarasenkova, N. (Ed). (2016). ***Conceptual framework for improving the mathematical training of young people:*** *Monograph*. In L. Kyba (A. Ed.). Budapest, Hungary: SCASPEE.

The authors:
I. Akulenko, O. Bochko, I. Bogatyrova, M. Bosovskiy, A. Bozhko, H. Brezhneva, S. Chernajeva, M. Donets, L. Golodiuk, O. Kolomiets, I. Lovyanova, G. Lutsenko, V. Milloushev, D. Millousheva-Boikina, Z. Serdiuk, N. Tarasenkova, O. Terekh, V. Tereshchenko, M. Tretyak, I. Vasilenko, & I. Volodko

Scholarly editor: Dr. habil., Full Prof. N. Tarasenkova
Author's Editor: L. Kyba

ISBN 978-963-12-7666-4
SCASPEE, Budapest, 2016

This book will be of interest to all researchers in the field of didactics of mathematics.

SCIENCE and EDUCATION a NEW DIMENSION

PEDAGOGY AND PSYCHOLOGY

www.seanewdim.com

THE JOURNAL IS INDEXED IN:

INDEX COPERNICUS: ICV 2014: 70.95; ICV 2015: 80.87

GLOBAL IMPACT FACTOR (GIF): 2013: 0.545; 2014: 0.676; 2015: 0.787

INNO SPACE SCIENTIFIC JOURNAL IMPACT FACTOR: 2013: 2.642; 2014: 4,685

ISI (INTERNATIONAL SCIENTIFIC INDEXING) IMPACT FACTOR: 2013: 0.465; 2014: 1.215

GOOGLE SCHOLAR

DIRECTORY OF RESEARCH JOURNAL INDEXING

ULRICHS WEB GLOBAL SERIALS DIRECTORY

UNION OF INTERNATIONAL ASSOCIATIONS YEARBOOK

SCRIBD

ACADEMIA.EDU

Cite this book:

APA Tarasenkova, N. (Ed). (2016). *Conceptual framework for improving the mathematical training of young people: Monograph*. In L. Kyba (A. Ed.). Budapest, Hungary: SCASPEE.

SSU Conceptual framework for improving the mathematical training of young people : [monograph] / Akulenko I., Bochko O., Bogatyrova I., Bosovskiy M., Bozhko A., Brezhneva H., Chernajeva S., Donets M., Golodiuk L., Kolomiets O., Lovyanova I., Lutsenko G., Milloushev V., Millousheva-Boikina D., Serdiuk Z., Tarasenkova N., Terekh O., Tereshchenko V., Tretyak M., Vasilenko I., & Volodko I.; Eds. prof. N. Tarasenkova, & L. Kyba. – Budapest : SCASPEE, 2016. – 212 p.

CONTENTS

PREFACE

Today the entire world is in constant search for up-to-date and democratic methods of the organization of education system enabling the increase of its efficiency, reduction of expenses, ensuring quick updating of the learning content pursuant to the requirements of the labor market and general education requirements. Therefore, Ukraine's movement towards the European educational space necessitates addressing a range of problems related to the gradual reform of the national secondary and higher education and the creation of high-quality education system that would meet international standards. Under these conditions there also increases the importance of exploring theoretical and practical experience of European countries that have reformed or are currently reforming their systems of education.

Solving the problem of improving the quality of basic mathematical training of high school and university students can be based on a coordinated implementation of systematic, praxeological, person-oriented, competency-based, and semiotic approaches. This requires the following: an in-depth analysis of the available data on the status of mathematics education in Ukraine and the scientific and methodological support of educational process in mathematics in high schools and universities; the development of the concept and the appropriate models of the improvement of mathematics preparation of students in terms of European integration and based on European experience and the principles of sustainable development; creating a balanced didactic support for mathematics education in high schools and universities, which would be aimed at the formation and development of mathematical competencies, intellectual skills, and productive thinking; the introduction of technologies of praxeological and intensive computer-based training; improving the system of methodical preparation of mathematics education students and mathematical sciences university teachers; the development of the system of methods of instrumental and statistical analysis of mathematical education.

The research in these areas by its results may be relevant to several related industries, as it may be the fundamental basis for improving not only the methods of teaching mathematics in primary and specialized schools and in higher education, but also the teaching methods of other natural and mathematical sciences (physics, chemistry, biology, etc.). The results can be used as an empirical basis for further theoretical generalizations in educational psychology, physiology and didactics of mathematics.

Equally important is the fact that improving the quality of mathematics education in Ukraine in terms of European integration processes can and should be aimed at fostering patriotism, a sense of pride in the country, and a relentless aspiration for self-actualization with the idea of further contributing to the development of welfare of Ukraine and each of its citizens.

Acknowledgements

The work was supported by the Ministry of Education and Science of Ukraine (state registration number 0115U000639).

The monograph is published by the support of Society for Cultural and Scientific Progress in Central and Eastern Europe, Budapest, 2016

CHAPTER ONE

THEORETICAL FRAMEWORKS

1.1. Cognitive Performance of Students in Mathematics Learning: Philosophy and Didactics

N. Tarasenkova

1. **Introduction.** The study of mathematics at school is of a paramount importance for the formation and development of the learners' personalities, shaping of their overall cultural competence, in particular – the ability to convincingly and consistently reason, compare and correlate, deduce consequences of a set of data, to perform simple calculations, and so on. At the same time, mathematics was and still is one of the most difficult subjects in the school curriculum. School curriculum includes the study of a broader scope of mathematical content, far beyond the daily needs of the students. At the same time the extracurricular life of a teenager is filled with more and more exciting new flow of information not related to mathematics and its strict order. Experience shows that motivating students to active and concerned assessment of the program material in mathematics is becoming more and more challenging. Hence, the scientists in the field of didactics of mathematics should address the concept of activity in the students' cognitive performance, find out its essence, and identify activity levels. This will help build a new strategy and tactics of mathematical training. Another thing which is to be mentioned is that in each classroom there are students with different levels of training and different ability to learn (educatibility), conventionally they are called quick, average and slow learners. In this regard, there are some reasonable questions: a) Is there any relationship between the degree of students' preparedness and the nature of their performance? b) What is the dynamics of each group's cognitive performance? c) Can cognitive performance of each group ascend its highest level within the same learning environment with the other groups? In this article, we do not claim to be exhaustive when looking for the answers to these questions. Rather, the answers should be considered as a pretext for launching a profound discussion in the press by the specialists in didactics of mathematics.

2. Theoretical framework. Such fundamental postulates of epistemology as: the material conditionality of knowledge and its social nature; the unity of dialectics, logic, and epistemology; the unity of theory and practice; the thesis about the individual's cognition of the world being connected with the world-historical process of development of knowledge (principle of historicism); and the activity of human cognition as a specific form of the reflection of reality are fundamental to the development of theories of teaching and education, and appropriate teaching systems[1] [2] [3] [4] (Hegel, 1929; Kremen' & Il'in, 2005; Russell, 1948; Sheptulin, 1967; etc.).

2.1. Reflection. Performance. Activity. Reflection as a general property and attribute of matter lies in the ability of matter to develop certain internal states in response to external stimuli, and to reproduce through these states the characteristics of external influence. The process of reflection lies in restructuring and transforming the

[1]Hegel, Georg. (1929). *The Science of Logic* : translated by W. H. Johnston and L. G. Struthers. London: George Allen & Unwin.

[2]Kremen', V. G., & Il'in, V. V. (2005). *Philosophy: thinkers, ideas and concepts. Textbook.* Kyiv: Kniga. (in Ukr.).

[3]Russell, B. (1948). *Human Knowledge, Its Scope and Limits*. London: George Allen & Unvin.

[4]Sheptulin, A. P. (1967). *The system of dialectic categories*. Moscow: Nauka. (in Rus.).

internal characteristics of the object as to counteract and compensate outside influences by replicating these influences. A product of reflection is a form of the rapport of the object with its environment, ensuring a comparatively stable existence of the object in new conditions reflected by this object.

The human mind and consciousness arising during a special type of interaction that connects people with the world are the highest forms of reflection. This type of interaction (rather – human impact on the world), which is called performance (activity) (Leont'ev[5], 1975), on the one hand, is the main form of the active relation of man to reality, on the other hand – it creates new, increasingly higher levels of activity.

Activity is among the characteristics of human performance and expresses the capacity for self-development and self-motion through the effectuation of reality-transforming creative actions. Goals and means of performance are not entirely spontaneous. The source events are far removed in time and space, they originate from a wide context of life, the essence of which is formed by inter-human relations. This is the philosophical principle of performance (activities). Philosophy[6] (Kagan, 1974) considers that, based on the activity principle, one can challenge the approach to treating human beings as such who have to adapt to the environment; it is also possible to properly evaluate the transformative and creative nature of human activity.

The diversity of the concept of human activity is determined by the fact that the very generic essence of man finds its expression in the performance. Philosophy provides the following characteristics of human performance: sociality, objectivity, appropriateness, awareness, mediation, and productivity. The main components of human performance are: the subject, the object and the activity itself, which results in a particular way of mastering the object by the subject or in establishing the communicative interaction with others by the subject. A particular individual, a social group or the society in general can be the subjects of performance. A natural object, an object of culture, a social institution, an ideal object, another person, and the subject itself can be the objects of performance. In the latter case the subject and the object of performance concur, there happens a kind of bifurcation of the subject of performance, which is typical of self-cognition or self-transformation.

Based on fixing one of the three main components of performance, philosophy distinguishes such basic human activities as transformative, cognitive, value-orientational, and communicative as well as artistic activities which perform the role of the organic synthesis of all four basic types. The philosophical analysis of cognitive activity is of particular interest to us.

2.2. Cognitive performance. Philosophy views knowledge as a special activity of reflection. In cognitive activity the object-directed performance of the subject does not change it, but only reflects and makes replicas. Thus, cognitive performance is aimed at developing an adequate image of reality. Proper consistent understanding of cognitive (reflective) performance is possible only through the recognition of practical transformational performance, but not the cognitive attitude to the world, to be the primary one. The said above nature of cognition reveals the following aspects of its performance.

1. The process of cognition, even of the most abstract one, never fully breaks away from the practical effect on real objects and their transformations in the course of intellection.

2. Products of intellection (cognition, learning) direct and regulate the subject's practical activities, providing their focused character, in particular – by predicting the results and selecting among them those that are required by the subject.

3. Cognition (intellection) itself acquires the performative character with all its specific features.

[5]Leont'ev, A. N. (1975). *Activity, Consciousness, and Personality*. Moscow: Politizdat. (in Rus.).
[6]Kagan, M. S. (1974). *Human activities: Experience of System Analysis*. Moscow: Politizdat. (in Rus.).

4. Cognitive activity can and does act as a "substitute" of practical activity not only for an individual but for the society in general. According to A. Leont'ev[7] (1975), the activity of cognition viewed in this aspect, is expressed in human ability to rely on a broad social experience, universal practice, and to overcome the narrowness of a person's practical interaction with the world and the limitations of their experience. It is through knowledge and the acquisition of the jointly produced means of activity that practice becomes a part of the individual's performance. The energy of cognition based on the assimilation of the results of social practice is capable of overcoming the inadequacy of the results of individual cognitive or practical actions and correcting them in accordance with the principle of plausibility.

5. Since the rules and means of learning are of social nature and are absorbed by the subject as something given and something that objectively exists, and the very need of their acquisition is also of social nature, then the highest expression of the activity of cognitive reflection is its social determination as opposed to external influences and states of the animal organisms. The active nature of cognitive reflection is also related with such its features as selectivity, future orientation (anticipatory reflection), the presence of motoric components and a system of its own resources of energy, the subordination of the cognitive performance to some certain previous plan, the mismatch of the cognitive image with its source or, conversely, their complete adequacy. The result, in the form of some kind of prior knowledge, may be given to the subject as incomplete, probabilistic, and schematic knowledge that acquires its specification in the course of cognitive activity.

From the position of general philosophical understanding of the nature and structure of knowledge [8] [9] [10] (Hegel, 1929; Kremen' & Il'in, 2005; Russell, 1948), the background knowledge is accessible to a human through sensory perception – feelings, perceptions, and representations. Rational cognition (thinking) is not limited to a simple summation or mechanical transformation of these sensations. The results of intellectual activity do not only provide new knowledge that is not directly contained in sensory data, but also directly affect the structure and content of knowledge. Herewith, the empirical data handled by scientific knowledge, are formed as a result of employing theoretical propositions for describing the content of sensory experience and allow for some theoretical idealization. Theoretical thinking envisages the ascent from the abstract to the concrete. Along with this sensual experience is understood not as a passive reflection and imprinting but as a moment of active practical, sensory and object-focused activity.

In cognition, categories and laws of dialectics are the forms of reflection of objective reality. Dialectics has it that in the objective reality everything is interconnected, that is – if phenomena and their aspects being in their universal interaction interpenetrate and, under certain conditions, merge, then the concepts through which a person learns the world, should inevitably be together in a strictly defined logical relationship, they should be flexible and this flexibility may result in the transition of some concepts into others and to the identity of opposites[11] (Sheptulin, 1967).

Thus, in managing education the cognitive performance of students should be based on the principles of dialectic knowledge, considering all its laws and patterns.

2.3. Educational and cognitive activities. As it has already been mentioned, cognitive activity represents one of the basic human activities, alongside with such as transformative, evaluative and communicative ones. Its peculiarity lies in the fact that

[7]Leont'ev, A. N. (1975). *Activity, Consciousness, and Personality*. Moscow: Politizdat. (in Rus.).

[8]Hegel, Georg. (1929). *The Science of Logic* : translated by W. H. Johnston and L. G. Struthers. London: George Allen & Unwin.

[9]Kremen', V. G., & Il'in, V. V. (2005). *Philosophy: thinkers, ideas and concepts. Textbook.* Kyiv: Kniga (in Ukr.).

[10]Russell, B. (1948). *Human Knowledge, Its Scope and Limits*. London: George Allen & Unvin.

[11]Sheptulin, A. P. (1967). *The system of dialectic categories*. Moscow: Nauka. (in Rus.).

the activity of the subject merely reflects (copies) the object to which it is applied, but does not change it.

Learning is both – reflective and transformative activity, as it is aimed at transforming the student's personal experience and their development by means of cognition and self-cognition. Cognitive and transformative components of this activity are inseparable and interdependent[12] (Tarasenkova, 1991).

The transformative nature of learning is connected with the nature of the student's performance as the student is the subject of activity. The activity serves as the inner regulator of educational and cognitive activity and is disclosed under self-progression, self-regulation, and self-actualization of a student's personality, that is, it is determined by the domineering of internal conditions over the external ones.

2.4. Components of cognitive activity. The study of the structure of the activity has been in the focus of close attention and research performed **by psychologists and educationalists.** In this article, we do not pursue the aim of comparing different points of view on the component composition of this phenomenon. We shall mention only the most obvious of its components[13] (Shamova, 1982).

Motivational component of cognitive performance reflects structural constituents of an individual's sphere of needs and motivations, among which the needs, interests, motives, and goals of cognitive activity are the most prominent ones.

Content-operational component is associated with the direct implementation of cognitive activity, its substantive basis, and the tools and methods of implementation.

Volitional component of cognitive performance is its attribute, as it reflects the impossibility of learning without the student's willpower, which has its own specific features and is dependent on many factors, not only the subjective ones.

2.5. Treatment of the performance objectives. In teaching, the teaching of mathematics including, the purpose of learning and cognitive activity is given to students from outside through the curriculum and textbooks[14 15 16 17 18 19 20 21] (Burda & Tarasenkova, 2007; Burda & Tarasenkova, 2008; Burda & Tarasenkova, 2009; Burda & Tarasenkova, 2010; Burda, Tarasenkova, Bogatyreva, Kolomiets, & Serdiuk 2013; Tarasenkova, Bogatyreva, Bochko, Kolomiets, & Serdiuk, 2013; Tarasenkova, Bogatyreva, Kolomiets, & Serdiuk, 2014; Tarasenkova, Bogatyreva, Kolomiets, & Serdiuk, 2015; etc.). However, a student may accept or not accept this goal, take it as an external necessity (emergency) or as an internally necessary (personally meaningful) objective. In addition, this purpose as an image of the final product of the upcoming activity may be attached both – situational and prospective significance. As a conscious objective determines the character of human performance, the student's attitude towards the performance objectives determines the implementation of each component of the

[12]Tarasenkova, N. A. (1991). *Activation of cognitive activity of students in a lecture and practical mathematics learning in school* (Doctoral dissertation). Kiev state pedagogical university, Kiev. (in Rus.).

[13]Shamova, T. I. (1982). *Enhancing school teaching*. Moscow: Pedagogika. (in Rus.).

[14]Burda, M. I., & Tarasenkova, N.A. (2007). *Geometry. Textbook for the 7th grade of the secondary school.* Kyiv: Publishing House "Osvita". (in Ukr.).

[15]Burda, M. I., & Tarasenkova, N. A. (2008). *Geometry. Textbook for the 8th grade of the secondary school.* Kyiv: Publishing House "Osvita". (in Ukr.).

[16]Burda, M. I., & Tarasenkova, N. A. (2009). *Geometry. Textbook for the 9th grade of the secondary school.* Kyiv: Publishing House "Osvita". (in Ukr.).

[17]Burda, M. I., & Tarasenkova, N. A. (2010). *Geometry. Textbook for the 10th grade of the secondary school.* Kyiv: Publishing House "Osvita". (in Ukr.).

[18]Burda, M. I., Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M., & Serdiuk, Z. O. (2013). *Geometry. Textbook for the 11th grade of the secondary school.* Kyiv: Publishing House "Osvita". (in Ukr.).

[19]Tarasenkova, N. A, Bogatyreva, I. M., Bochko, O. P., Kolomiets, O. M., & Serdiuk, Z. O. (2013). *Mathematics*: *textbook for the 5th form for the secondary schools*. Kyiv: Publishing House "Osvita". (in Ukr.).

[20]Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M., & Serdiuk, Z. O. (2014). *Mathematics*: *textbook for the 6th form for the secondary schools.* Kyiv: Publishing House "Osvita". (in Ukr.).

[21]Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M., & Serdiuk, Z. O. (2015). *Algebra*: *textbook for the 7th form for the secondary schools*, Kyiv: Publishing House "Osvita". (in Ukr.).

activities (motivational, meaningful, operational, and volitional) and of the performance as a whole.

The attitude of schoolchildren to such school subject as mathematics is ambiguous because of the subject's specificity[22 23 24 25 26] (Tarasenkova, 2013; Tarasenkova, 2014; Tarasenkova, 2015; Tarasenkova, Bogatyreva, Bochko, Kolomiets, & Serdiuk, 2013; Tarasenkova, Bogatyreva, Kolomiets, & Serdiuk, 2015). Understanding the general developmental importance of this subject, most students accept principal objectives of studying mathematics; at the same time, some of them may not accept any specific objectives (for example, the task to explore different ways of solving quadratic equations).

2.6. Motives of upcoming activities. In the process of realizing the goal of studying the specific content of mathematics and evaluating the ways of their achieving, the learners are having the motives for the upcoming activities formed. These motives manifest the learners' needs and interests (Bogoyavlenskaya, 1983). Considering the data of psychological science about the nature of the motivation for studying and learning, we shall group them in the following way[27] (Tarasenkova, 1991):

1) motives of duty – the goal is forced in its character, the product of activity is not evaluated in terms of its significance for self-development;

2) motives of personal development – the goal is taken as being of inner necessity, but the product of learning is evaluated only as situationally meaningful for the student, it is regarded as a necessary means to meet the needs of self-assertion rather than the needs for self-knowledge and self-development;

3) cognitive motives – the purpose of learning is of personal importance for the learner, the product of the upcoming activities is evaluated as promisingly meaningful, that is, as requisite for self-knowledge and self-development and important to such an extent that there arises the necessity to save it for future active use.

2.7. Behavioral Activity. In teaching mathematics, if the goal of educational and cognitive activity is not realized, and therefore is not accepted, or it is recognized but is not accepted, we do not deal with learning as a purposeful process of transformation of the student's personal experience, but we have to do with the student's impulsive behavior, aimed at "removing" the discomfort created by external and unacceptable conditions for them.

In situation like this the student displays activity which can be manifested in behavior which only bears resemblance to educational and cognitive activity, or it may acquire the forms of antisocial behavior. In contrast to the teaching and learning activity, this one can be qualified as behavioral[28] (Tarasenkova, 1991).

2.8. Assessment of the achievability of goals. Acceptance or rejection of the goals by the students is preceded by the assessment of the possibility of achieving it; this assessment

---

[22]Tarasenkova, N. (2013). The quality of mathematical education in the context of Semiotics. *American Journal of Educational Research, 1*(11), 464-471. http://doi: 10.12691/education-1-11-2.

[23]Tarasenkova, N. (2014). Peculiar Features of Verbal Formulations in School Mathematics. *Global Journal of Human-Social science : G : Linguistics & Education*, *14*(3), 61-67. http://globaljournals.org/journals/human-social-science/g-linguistics-education

[24]Tarasenkova, N. (2015). Non-verbal Shells of the Instructional Mathematical Content. *American Journal of Educational Research*, *3*(12B), 1-5. http://doi: 10.12691/education-3-12B-1.

[25]Tarasenkova, N. A, Bogatyreva, I. M., Bochko, O. P., Kolomiets, O. M., & Serdiuk, Z. O. (2013). Conceptual principles of development of text-books on mathematics for 5–6 classes. *Science and education a new dimension*, *2* (Marhc), 34-38. (in Ukr.). http://seanewdim.com/published-issues.html

[26]Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M., & Serdiuk, Z. O. (2015). The structure and content of teaching kits on algebra for the 7th Form. *Science and education a new dimension*, *3*(50), 12-18. (in Ukr.). http://seanewdim.com/published-issues.html

[27]Tarasenkova, N. A. (1991). *Activation of cognitive activity of students in a lecture and practical mathematics learning in school* (Doctoral dissertation). Kiev state pedagogical university, Kiev. (in Rus.).

[28]Tarasenkova, N. A. (1991). *Activation of cognitive activity of students in a lecture and practical mathematics learning in school* (Doctoral dissertation). Kiev state pedagogical university, Kiev. (in Rus.).

is realized as if on two levels: the preliminary and the detailed ones[29] (Tarasenkova, 1991). The preliminary assessment (estimation) of the achievability of goals is an unconscious act. It rests on the instant comparison of, on the one hand, the student's idea of the necessary conditions for achieving the goal (this idea may supposedly be incomplete or incorrect) and, on the other hand, their ideas of their own ability (this idea may possibly be inadequate). The result of a preliminary assessment is the orientation (positive or negative), which manifests itself in the student's attitude to learning. Negative orientation is one of the sources of the occurrence of the behavioral activity.

The detailed assessment of the achievability of the goals takes place while isolating and understanding the necessary conditions and possible ways of its achieving, and comparing them with the real or potential possibilities of the student. The underestimation of the potential possibilities leads to the rejection of the goals and, as a consequence, to the manifestations of behavioral activity.

Another source of such activity of the student may appear when the high school student, having initially overestimated their capabilities, in the course of work on mathematical instructional material, loses confidence in the attainability of the goal of learning and abandons it.

2.9. Motivations and cognitive performance. If the student recognizes and accepts the goals of learning mathematics as achievable, then their educational and cognitive activity is stimulated by complex motivations, which include those represented above. The domineering of the motives in this or that group generates a corresponding level of cognitive performance[30] (Tarasenkova, 1991).

Thus, the motives of duty, being external to the purpose of the activity, bring forth **the stimulus-productive level** of the students' activity. These students are to be constantly compelled to study math. The motives of personal success are also external as related to the purpose of the activity, and also generate a stimulus-productive level of activity. However, to meet the needs corresponding to these motives, the student makes use of the tools of self-awareness and self-development, thereby consciously they change their experience being guided by cognitive motives, though do not care about the long-term preservation of these new formations for further use. In other words, under the domineering of the motives of personal success, the student's performance at certain moments acquires the characteristic features of either heuristic or the stimulus-productive activity. To adequately reflect the relationship between the term and the essence of the activity stimulated by the motives of personal success, this level of activity in the study of mathematics should be called **a situational-heuristic one**.

If the student's activity, when studying mathematics, is stimulated by cognitive motives, the student's performance may manifest itself on a heuristic, as well as on the highest, that is, creative, level. It should be borne in mind that the creative level of the student's activity when studying mathematics, differs greatly from the same level of the researcher's performance. A scientist chooses the object of study and research on their own; they determine the goal of their performance. A high-school student obtains the object of knowledge and cognition through the preassigned content of mathematical education, thus having the goal of their activities set. However, the student may need to deepen and broaden their understanding of the object being studied. This will cause the necessity of complementing the goal set by new personal goals, which correspond to the creative level of performance in the fullest sense of the meaning of the word "creative". To differentiate the activities and cognitive performance of the scientist and the student at this level, the activity of the student should be called instructional-and-creative level

[29]Tarasenkova, N. A. (1991). *Activation of cognitive activity of students in a lecture and practical mathematics learning in school* (Doctoral dissertation). Kiev state pedagogical university, Kiev. (in Rus.).
[30]Tarasenkova, N. A. (1991). *Activation of cognitive activity of students in a lecture and practical mathematics learning in school* (Doctoral dissertation). Kiev state pedagogical university, Kiev. (in Rus.).

of performance. This level of activity is connected with the **educational and cognitive goal setting**.

2.10. Factors of difference between the activity levels. The essential difference between the activities of the student on the heuristic and educational-creative levels of activity is manifested in the attitude of the students to the contents of the product of their learning activities [31] (Tarasenkova, 1991). In the first case the content (its volume) is treated as necessary and sufficient, in the second – as necessary, but not sufficient.

The essence of the difference between the stimulus-productive and situational-heuristics levels of performance is expressed in the attitude to the quality of the product of learning activities (the degree of its correspondence to the standard). This attitude may be indifferent or non-indifferent.

The difference between situational-heuristics and heuristic levels of teaching mathematics is manifested in the attitude of the student to the necessity to make a product correction (to bring it to compliance with the standard). The attitude of the student may be negative – as to the imposed act, or positive, as to a necessary act.

In the general form, the aggregated data are presented in Table 1.

Table 1. Activity levels

Level of activity	Domineering motives	Manifestations		
		The content of the produce is assessed as	Attitude to the quality of the produce	Attitude to the correction
stimulus-productive	of duty	necessary and sufficient	indifferent	negative, as to the imposed act
situational-heuristic	of personal success	necessary and sufficient	non-indifferent	negative, as to the imposed act
heuristic	cognitive	necessary and sufficient	non-indifferent	positive, as to the necessary act
educational-creative	educational-cognitive goal setting	necessary but insufficient	non-indifferent	positive, as to the necessary act

3. Results of observations. Analysis of the experience of mathematics teachers has allowed of establishing the following[32] (Tarasenkova, 1991):

1. The relationship between the degree of preparedness of pupils and the nature of their activity is manifested only in the case if the teacher provides training on a strictly defined level of difficulty.

2. The level of the difficulties of training offered by the teacher is mistakenly interpreted as the focus, when choosing the content, forms and methods of work, on the cognitive capabilities of the corresponding group and training all students at this level.

3. Direct relationship between students' preparedness and activity is observed only when the teacher focuses only on the abilities of smart students.

4. Most teachers tend to conduct training on "the highest level of difficulties". In this case we observe the following:

[31]Tarasenkova, N. A. (1991). *Activation of cognitive activity of students in a lecture and practical mathematics learning in school* (Doctoral dissertation). Kiev state pedagogical university, Kiev. (in Rus.).

[32]Tarasenkova, N. A. (1991). *Activation of cognitive activity of students in a lecture and practical mathematics learning in school* (Doctoral dissertation). Kiev state pedagogical university, Kiev. (in Rus.).

a) representatives of different groups have different attitude to the planned minimal results of studying the topic;

b) this attitude manifests itself in different ways depending at which point of the learning process the planned minimal results are made known to the students; it can have a significant impact on the increase in the activity of average students and minor impact on quick and slow learners;

c) the accessibility of the instruction and the level of the comprehension of the material after some additional instruction (if) have a dominant influence on the nature of the activity of high school students;

d) there is the effect of "stunness" after the presentation of new material (especially in large blocks) in the average and slow learners, which partly disappears in the average students after working on the tasks of the mandatory level (if this kind of work takes place) and does not disappear in the slow learners throughout studying the whole topic;

e) the use of differentiated array of tasks and individual assignments significantly increases the activity of quick learners, moderately – of the average and slightly – of the slow ones.

All this can be explained by the fact that the presentation of new material at the "high level of complexity" indirectly forms the idea of the goals of studying the themes that are much more complicated than the minimum and, therefore, are unattainable. At the same time, if the minimal results are achieved, the students still retain a strong feeling of discontent that inhibits rather than stimulates their activity.

Withal, if the teachers in their instructional performance use the method of accessibility and openness of the learning objectives, this pattern is not observed. To achieve this the goals of mastering the theme are differentiated considering the abilities of each group of students; they are communicated in various ways to students; during the presentation of the new material the content is clearly delineated given the goals; each of the groups is informed of the work plan on the theme; the teacher sees to it that each student should understand the new material and acquire the relevant skills. Such training tactics allows teachers to maintain high activity of all learners throughout the period of studying the topic. There are no facts of behavioral activity. All students display concernment about the quality of their results and treat the correction as a necessary act. All students independently and consciously choose the difficulty level of learning, i.e., recognize the elected amount of content as necessary and sufficient.

There have been cases when some students changed the chosen level. Thus, not only average but also slow students moved to a higher level. This transition actually means that the volume of the learning outcome was recognized by the students as sufficient, and the learning goals and objectives were complemented by new ones which were put forward by the students themselves.

4. Discussion. Learning is a poly-motivated process and the domineering of the motives of this or that group is not absolute and is not constant, that is, at any stage of the learning process there may take place the motivational reorientation, depending on the attitude of the learner to the purpose of the activity at this very moment. Therefore, the manifestation of activism at different levels is a dynamic process, i.e., the transition from a lower to a higher level of activity and vice versa is possible, up to the manifestation of behavioral activity.

Behavioral activity is a method used by a learner to protect themselves from unwanted expenditure of internal resources. This activity is not a manifestation of cognitive performance, therefore, a priori; it cannot lead to positive learning outcomes. In this regard, one of the first requirements, the organization of the process of teaching mathematics should be subordinated to, is connected with the necessity of creating such objective conditions that would exclude the possibility of behavioral activity of the students.

The second requirement to the organization of teaching mathematics reflects the necessity to create the objective conditions that preclude the possibility of moving the student's activity from a higher to a lower level and enable the serial transfer of the student's activity to its highest level.

Compliance with this requirement is nothing else but the procuring of the objective prerequisites for the implementation of the principle of activism in learning. This principle presupposes not only external (objective) conditions created by the teacher, but also the internal (subjective) conditions specific to the student and demands the prevalence of internal conditions over the external ones.

Due to the fact that the limited facilities of the teachers do not allow them only through their own efforts to ensure this predominance, there arises the need of stating one more requirement to the organization of teaching mathematics. Its essence lies in the fact that the external conditions should provide an opportunity for students (after some training) to set the goals on their own, to perform planning, to self-organize and to self-control learning and cognitive activity.

5. Conclusions. Philosophy, psychology and math didactics is a scientific basis of the activation of the cognitive performance of students in learning mathematics. The philosophical understanding of the nature of cognitive performance is the basis for the development of teaching theories and teaching practices. The psychological principles of initiative, self-organization, development, teamwork, role-based participation, accountability, psychological support, as well as didactic principles, among which the principle of consciousness in learning and the principle of accessibility are of special significance, they make up the psycho-pedagogical framework for the implementation of the requirements to the arrangement of the learning process, which, in its turn, provides the purposeful positive impact on the nature of activism in the students' cognitive performance.

Theoretical analysis and observations allowed us to put forward the assumption that, regardless of the level of preparedness, the activity of the students can be manifested at the highest, educational and creative level, but only if the training takes place in the zone of "proximal development", which is individual for each student.

1.2. Educability, learning attainment, and competence of the students in the context of mathematical education

O. Bochko

Specifics of teaching elementary mathematics require the consideration of such personal characteristics of individual students as educability and learning attainment that influence the formation of the students' competence in elementary mathematics.

The term "educability" is widely used in psychological and pedagogical literature. In particular, A. Markova, T. Matis, & A. Orlov[1] (1990) says that educability is defined as a student's ability to assimilate new knowledge and new ways of obtaining them, and the readiness to move to new levels of mental development.

In the works of Z. Kalmykova[2] (1981) it is indicated that educability is a system of an intelligent personality traits, basic qualities of mind that affect the quality of training activities (plus the presence of the original minimum knowledge, positive motivation, etc.).

The main indicators should include training *dynamics* in the process of learning and *formation* of the skills, *ease* of the assimilation (absence of tension, fatigue, and feeling pleasure from obtaining knowledge), *flexibility* in the ability to switch to new methods and techniques, long and strong *retaining of the* material learned. Z. Kalmykova[2] (1981) considers brevity and tempo of thinking; a specific amount of the material that can be processed; the results achieved; a self-phased process of solving specific problems; the limited dose of help; the time spent on solving problems; the ability to self-training; performance characteristics, and endurance to be the key features of educability.

Noteworthy are the educability characteristics highlighted by A. Markova, T. Matis, & A. Orlov[1] (1990): orientating at new conditions; initiative in the choice of optional tasks, self –help in doing more complex exercises.

D. Bogoyavlenska[3] (1983) notes that these figures may correlate with the concept of intellectual initiative as a phenomenon of creative activity, persistence in achieving goals and ability to work in a situation of obstacles, distractions; predisposition and readiness to accept others' help, absence of resistance.

According to the characteristics described above by Z. Kalmykova[2] (1981), the method of diagnostics of educability is based on the following conditions: diagnostics should be comprehensive, based on synthetic (not analytical) way.

Researchers are indicates the following features of educability: development of new concepts; making generalizations; flexibility of thinking; ability to solve problems in different ways; retaining general concepts; generalized knowledge; intellectual activity.

The higher educability is the faster and easier the person acquires new knowledge, the freer they operate in a relatively new environment, and the higher the rate of mental development is. It is clear that learning success, not less than intelligence, also affects such students' characteristics as attention, memory, motives, and traits like these. Educability is not necessarily an inherited quality; it may be the result of detailed, expanded explanations of the material. Educability is manifested in a relatively independent acquisition, opening new knowledge for themselves, the breadth of the transfer of knowledge to new situations while solving unusual and new problems.

We can identify those qualities of mind that are formed in students, and determine the level and specificity of educability. Here is a brief description of these qualities.

The *depth* of the mind can be detected through a number of significant signs that a person can abstract while mastering new material, and the level of their generalization.

---

[1]Markova, A., Matis, T., Orlov, A. (1990). *Forming learning motivation*. Moscow: Prosvieshcheniie. (In Rus.).
[2]Kalmykova, Z. (1981). *Productive thinking as education fundament*. Moscow. (In Rus.).
[3]Bogoyavlenskaya, D. (1983). *Intellectual activity as a problem of creativity. Monograph.* Rostov-na-Donu: Publishing house of Rostov University. (In Rus.).

The opposite quality – the *limitation* of the mind, which is manifested in distinguishing external, individual attributes by setting random connections between them.

Flexibility of the mind is traced in the variability of mental activity under changing conditions, dealing with different situations and solving problems. A person who has a flexible mindset easily switches from direct connections to the inverse, from operations within one system to another if it is required by the task. It may abandon the usual action to overcome the "barrier-setting prior experience" if an attempt to solve the problem based on it did not lead to success, and look for another way of solving the problem. *Inertia* of reasoning is the phenomenon under which the mental activity sticks to patterns, displays gradual implementation of habitual thoughts, difficulty in the transition from one action to another.

Firmness of mind. In order to successfully master new knowledge and to operate it, the student should not only highlight the essential features required by the situation, but keep in mind all of them and to act accordingly without exposing oneself to external and random features, which can lead to an erroneous interpretation. The instability of mind is manifested in finding it difficult to focus on the new features, be those new concepts or content patterns, to transit from one system to another and to act being influenced by random associations.

Independence (autonomy) of the mind is manifested in the active search for new knowledge and ways of solving problems when a person seeks not only for the right but also for the optimal solutions. *Inherited* mindset is observed in trying to copy the already known methods of solution, avoiding intellectual strain, being blind to errors.

So we enumerated the main characteristics of the mind, which are the building bricks of educability.

B. Krutetskyy[4] (1972) focuses on it that educability may differ from student to student. For the students with higher educability levels fast paced learning, which is associated with the ability to quickly summarize, high flexibility (mobility) of mental process, etc. are typical. Students with lower educability show a slow pace of assimilating knowledge, are uncapable of quick summarizing, the inertia of thinking is one of their common features.

As first-year students have different learning attainments when starting mathematical training, the acute problem is to prognosticate their educability and to didactically consider learning strategies.

Specificity of the combination of characteristics pertaining to educability and different levels of their manifestation form an individual picture (pattern) of educability of students. In creating a so-called "passport of educability" the first steps should be made towards the development of a student's individual educational trajectory.

To define the specifics of educability of the students of the mathematical faculties of universities we must take into account the specific learning content, including mastering the features of the objects and the requirements to the learning outcomes.

Studying mathematical disciplines we refer to concepts and definitions, math facts (axioms, theorems, and formulas), ways of working (algorithms, rules, heuristic schemes, methods of proof of statements and ways of solving certain classes of problems, etc.). N. Tarasenkova[5] (2002) points out that in the structure of the method can distinguish the *contents* (epistemological) and *operational* (activity) components.

The content of the component method is a system of knowledge, which includes: 1) the source of knowledge about the object and its properties; 2) the resulting knowledge, that is the results of operations performed; 3) knowledge of the operating mode of composition; 4) knowledge of subject-intellectual and practical tools, necessary to

[4]Krutetskiy, V. (1972). *Essentials of pedagogical psychology*. Moscow: Prosvieshcheniie. (In Rus.).

[5]Tarasenkowa, N. (2002). *Using sign and symbolic means in teaching mathematics. Monograph*. Cherkasy: "Vidlunnya-Plyus". (In Ukr.).

perform the activity; 5) reference system of selecting a particular way of many others.

The *operating mode* of the component is associated with the immediate implementation of its action.

Mastering the semantic component of the method is characterized by such neoplasms in their personal experience as knowledge and mastery of the operational components being reflected in skills. In studying mathematical sciences students learn theory and acquire practical skills. Among the latter, it is advisable to differentiate two groups – general and special skills.

The study defined that the peculiarities of the mathematical faculties students' educability affect specific content of training, including mastering features of objects (concepts and their definitions, math facts, and styles) and the corresponding requirements for learning outcomes, according to the three allocated educability types: *object* (when unfamiliar objects and methods of transformation on other objects are assimilated); *procedural* where familiar objects and methods of transformation get a new slant; *combined*, when objects and their transformation, as well as transformation methods are new to the student. Diagnostics of these types of educability is made through special terms and related problems.

Educability of students is reflected in their learning attainments. The hierarchy is as follows: mastering a particular object of assimilation; mastering a system of the objects of assimilation; mastering academic subjects; acquisition of a thematic module; mastery of the module course; successful mastery of the whole course.

These educational achievements are not stable; they can change over time, particularly as a result of forgetting what has been learnt. In addition, they display a certain didactic goal of the cycle: studying an individual object of learning, studying several objects of learning, mastery of a theme, studying the semantic module, the module of the course, and finishing the course of studies. Therefore, educational achievements of the student should be considered through the situational characteristic of the results of the studying process.

Educational results get fundamentally different when a student's knowledge is no longer actively used, but some learned material remains in a state close to active and these skills are restored quite quickly and easily. This system of knowledge is called residual knowledge. The ability of the student not only to remember, but also to apply this knowledge determines the *learning attainment*.

As I. Zimniaya[6] (1997) claims, a *learning attainment* is the most professional property of the individual graduate and it consists of professional knowledge areas, skills and abilities relating to the minimum content of the educational program with the following disciplines: general humanitarian and socio-economic disciplines, general maths, natural sciences, all professional, special and specialization-related disciplines, optional courses, electives, and internships.

Learning attainment includes the knowledge, skills and abilities acquired as a result of studying some subject and becoming the so-called residual knowledge. Thus, the structure of the subject of training can distinguish three components: knowledge, skills, and techniques. Studies have shown that the concept of "Learning attainment" is closely linked to the concept of "competence".

Such leading scientists as N. Bibik[7] (2008), O. Ovcharuk[8] (2003), O. Pometun[9] (2004), N. Tarasenkova, & V. Kirman[10] (2008) and others disclose the nature of the categories "competence" and "competency".

---

[6]Zimniaya, I. (1997). *Pedagogical psychology*. Rostov-na-Donu: Phoenix. (In Rus.).

[7]Bibik, N. (2008). Competence in teaching. *Encyclopedia of education* (Academy of Pedagogy of Ukraine, Ed. by Kremen', V. G.). Kyiv: Yurinkom Inter. (In Ukr.).

[8]Ovcharuk, O. (2003). Competencies as a Key to Educational Content Renewal. *Reform Strategy for Education in Ukraine: Education policy recommendations*. Kyiv: KIS. (In Ukr.).

According to O. Pometun[11] (2004), human competence is organized in certain sets of knowledge, skills, abilities, and attitudes, that enables a person to determine (identify) and solve problems regardless of the situation. Thus, competence acts effectively and it is the feature which is typical of education.

In Encyclopedia of Education[12] (Kremen', V. (Ed.), 2008) it is stated that competence in teaching (lat. *competentia – a range of issues in which the person is well understood*) is also acquired by means of non-formal education, under the influence of the environment etc.

In foreign sources learning attainment is often associated with such notions as "ability to", "set of skills", "craft", "readiness", "knowledge in action", "capacity" etc. In addition, competence in training is seen as an integrated result that involves a shift of emphasis from saving regulatory knowledge, skills, and abilities to the development of students' ability to act practically and apply the experience of successful activity in a particular area.

Countries where the competence approach to education has been implemented for a long time, there are common trends in the development of a system of competence in education, at different levels and in different meanings of the phrase, as a certain hierarchy of competencies in learning: *key* or *base*, these are based on cognitive processes and are found in different contexts (they can be presented as an "umbrella" over the whole learning process); *common* – these belong to a certain set of subjects or educational fields; they have a high degree of generality and complexity; *subject* –these are those that are acquired in the study of certain subjects.

The results of the working group of the Ukrainian scientists and practitioners[13 14 15] (Bibik, 2008; Ovcharuk, 2003; Pometun, 2004; and others) were the developed theoretical and practical issues of the implementation of competence approach in education in Ukraine. As a result there was suggested a list of key competencies in teaching, learning (ability to learn), civil, general cultural, informational, social, and health-care spheres which are detailed in a set of knowledge, skills, values, attitudes, abilities in different educational segments and spheres of students' life.

The notion of the students' competence in elementary mathematics has been put into practice, the contents and the structure of this competence have been explained, the criteria and indicators of its formation for the students of classical mathematics faculties has been developed. The methodology of determining the requirements to the level of the results of the learning practicum on solving mathematical problems has been developed; the contents have been structured; the methods and organizational forms of training in the course of the practicum on solving of mathematical problems have been selected and applied.

In accordance with the established competencies we distinguish the following types

[9]Pometun, O. (2004). Implementation of competence approach – a promising trend of modern education. *Bulletin of school exchanges. Integrating competence approach – perspective direction of modern education. 22.* Retrieved from http://visnyk.iatp.org.ua/visnyk/issue_article;22 (In Ukr.).

[10]Tarasenkova, N. A., & Kirman, V. K. (2008). The content and structure of mathematical competence of students of secondary schools. *Mathematics in school*, *6*, 3-9. (In Ukr.).

[11]Pometun, O. (2004). Implementation of competence approach – a promising trend of modern education. *Bulletin of school exchanges. Integrating competence approach – perspective direction of modern education. 22.* Retrieved from http://visnyk.iatp.org.ua/visnyk/issue_article;22 (In Ukr.).

[12]Kremen', V. (Ed.). (2008). *Encyclopedia of education* (Academy of Pedagogy of Ukraine). Kyiv: Yurinkom Inter. (In Ukr.).

[13]Bibik, N. (2008). Competence in teaching. *Encyclopedia of education* (Academy of Pedagogy of Ukraine, Ed. by Kremen', V. G.). Kyiv: Yurinkom Inter. (In Ukr.).

[14]Ovcharuk, O. (2003). Competencies as a Key to Educational Content Renewal. *Reform Strategy for Education in Ukraine: Education policy recommendations*. Kyiv: KIS. (In Ukr.).

[15]Pometun, O. (2004). Implementation of competence approach is a promising trend of modern education. *Bulletin of school exchanges. Integrating competence approach is a perspective direction of modern education. 22.* Retrieved from http://visnyk.iatp.org.ua/visnyk/issue_article;22 (In Ukr.).

of competence: *key* (mastery of the target level of educational content); *common* (mastery of certain subjects and educational fields); *substantive* (related to specific content). The list of competencies is related to the respective competences.

The similarity of such categories as "learning" and "competence" is that firstly, each of them is an integral factor characterizing a personality. Secondly, learning attainment and competence have such common structural components as knowledge, skills, and abilities. However, the term "learning attainment" is not identical to the concept of "competence". If a student is competent it characterizes them as having personal attitudes and performing specific activities of the subject.

Figure 1 shows our vision of the structure of competence in elementary mathematics (CEM)[16] (Volovyk (Bochko), 2010). CEM should be considered to be a set of integrated personality traits based on knowledge, skills, abilities, personal attitudes, and one's practical experience in elementary mathematics and the students' academic performance in this field. CEM is characterized by the presence of extensive and comprehensive knowledge of elementary math, proficiency in solving elementary mathematics problems of different complexity levels, mathematical style of thinking, high level of oral and written language in the subject area of elementary mathematics, mastery of written and oral mathematical reasoning, mathematical modeling skills, abilities to use mathematical instruments, efficiency in using information and communication technologies for solving problems in elementary mathematics, techniques of integrating the material from various sections of the elementary mathematics course (inner-disciplinary relations) and establishing the connections between the topics from the course material elementary mathematics and other fundamental mathematical disciplines (interdisciplinary communication). CEM consists of four components: praxeological, reflexive, and self-development. The criteria for forming each CEM component are: knowledge in elementary mathematics; ability to apply theoretical knowledge to solve elementary mathematics problems of different levels of complexity and solve problems from other fundamental disciplines of the mathematical cycle; capacity for reflection; positive motivation for self-improvement in the field of elementary mathematics.

In general, CEM should be both – basic and effective characteristic feature of the educational process at mathematical faculties of classical Universities.

[16]Volovyk (Bochko), O. (2010). *Methodological principles of organization and effectuation of practicum on solving mathematical problems in classical universities. Thesis.* Cherkasy: Bohdan Khmelnytsky National University of Cherkasy. (In Ukr.).

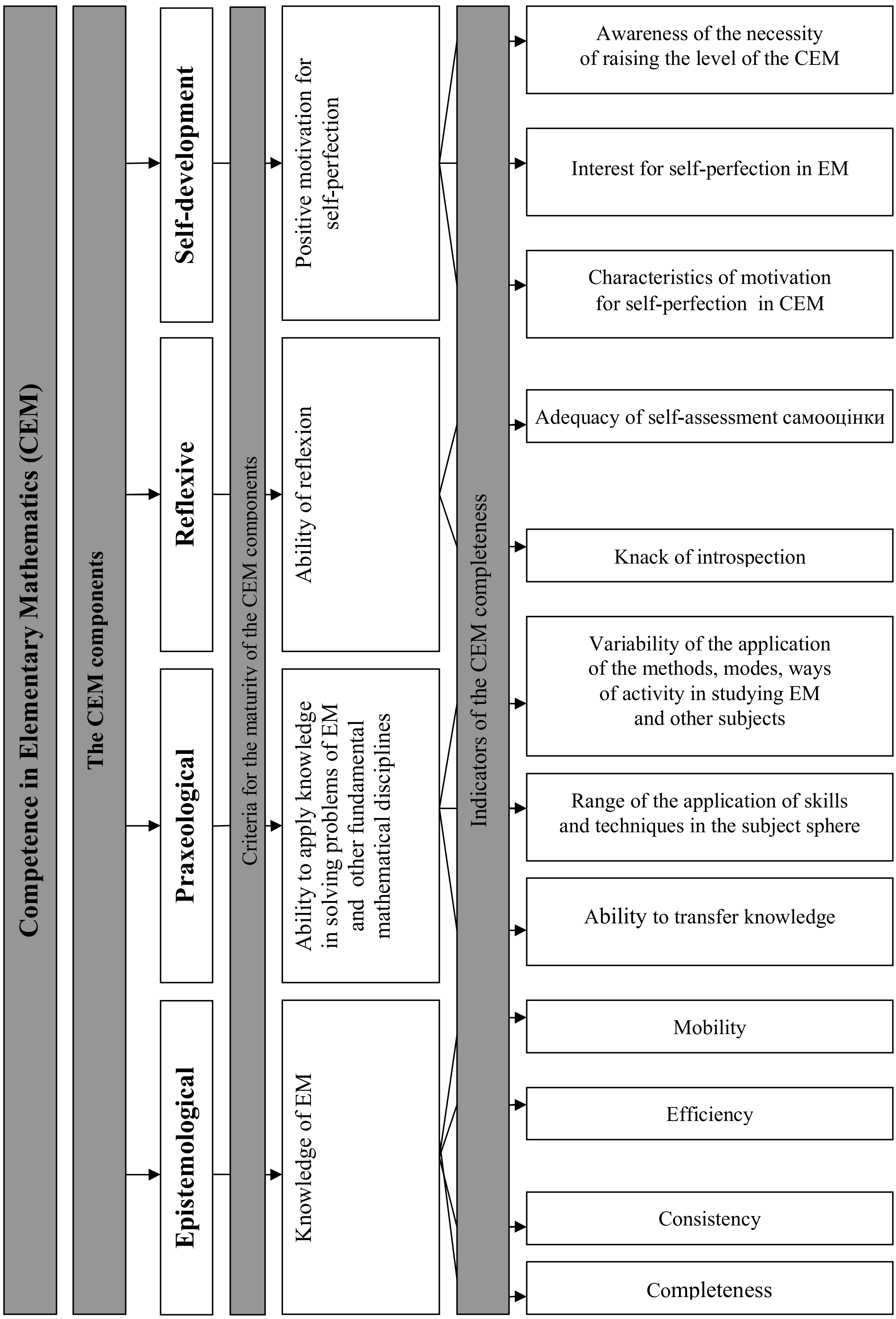


Figure 1. Structure of the competence in elementary mathematics

1.3. The Role of the Universities in the Development of the Mathematical Culture of Society

M. Tretyak

The main objective of the modern higher educational institutions is the training of highly skilled and competent specialists. National doctrine of the development of Ukrainian education maintains that fundamentalization, informatization, computerization, democracy, humanism, and openness are the main ways of such objectives achievement. The development of fundamental sciences is unavoidably connected with their mathematization and, therefore, the mathematization of fundamental education as a whole. Under such conditions high level of mathematical culture of a huge amount of experts and the society in general is becoming an extremely necessary and important feature.

Even without the detailed analysis of the term "mathematical culture of society" it should to be noted that the level of mathematical culture of society is determined by the level of mathematical culture of the following communities: scientific and technical, engineering, financial and economic, administrative, student etc. It seems reasonable that the leading role in the formation and development of the mathematical culture of the communities mentioned above is played by universities, chiefly – by classical and pedagogical ones. In this role, universities are facing serious challenges. Such famous thinkers as J. Ortega y Gasset[1] (2010) and M. C. Petrov[2] (2004) tried to apprehend such challenges and to suggest the ways of approaching them. In quite a number of studies the high purpose of the University as well as an ideal model of a modern University are considered. At the same time, the ways of the embodiment of theoretical ideas are not described clearly enough and are mostly declarative. Also, such modern famous pedagogues and theorists of higher education as B. S. Gershynskyy[3] (2002), A. I. Kuzminskyy[4] (2011), V. O. Rybin[5] (2012), focus on these problems. The studies of the scholars mentioned above are characterized by practical orientation and detailed development of the methodological approaches concerning practical implementation of the main ideas.

The issues connected with the formation of mathematical culture of students of different specialties, are very topical. Over a long period of time these issues have been investigated by a significant number of well-known scholars. Among them there are famous mathematicians: V. I. Arnold[6] (1997), B. V. Gnedenko[7] (2006), A. M. Kolmogorov[8] (1988), A. Poincare[9] (1983), O. Ya. Khinchin[10] (2013) et al. as well as pedagogues and psychologists: E. O. Lodatko[11] (2011), G. O. Mykhalin[12] (2003), S. O. Rozanova[13] (2003),

---

[1]Ortega y Gasset, J. (2010). *Mission of the university*. M. : Izd. dom. Gos. Universiteta – Vysshey shkoly ekonomiki. (In Rus.).

[2]Petrov, M. K. (2004). *History of the European cultural tradition and its issues*. M.: ROSSPEN. (In Rus.).

[3]Gershynskyy, B. S. (2002). *Philosophy of education in XXth century*. M.: Pedagogicheskoe obshchestvo Rossii. (In Rus.).

[4]Kuzminskyy, A. I. (2011). *University Pedagogy*. K.: Znannya. (in Ukr.).

[5]Rybin, V. A. (2012). *University of the XXIst century: Anthropological prospects of education and culture*. M.: Librocom. (In Rus.).

[6]Arnold, V. I. (1997). Mathematics and mathematical education in the modern world. *Mathematical Education*, *2*, 22–23. (In Rus.).

[7]Gnedenko, B. V. (2006). *On teaching mathematics at universities and pedagogical institutions at the turn of the trird millenium*. M.: ComKniga. (In Rus.).

[8]Kolmogorov, A. N. (1988). *Mathematics – science and profession*. (Ed. G. A. Galperin). M.: Nauka. Glavnaiya redaktsiya fiziko-matematicheskoy literatury. (In Rus.).

[9]Poincaré, H. (1983). *On Science*. M.: Nauka. Chief Editorial Board of Physics and Mathematics Literature. (In Rus.).

[10]Khinchin, O.Ya. (2013). *Education papers: Issues of teaching mathematics. Fighting methodical clichés*. (Ed. B.V. Gnedenko). M.: ComKniga. (In Rus.).

[11]Lodatko, Ye. O. (2011). *Mathematical culture of the primary school teacher*. Rivne-Slovyansk: Entrepreneur B. I. Matorin. (in Ukr.).

A. I. Kuzminskyy, N. A. Tarasenkova, & I. A. Akulenko[14] (2009) et al.

A large majority of the mathematical and pedagogical personnel are mathematics programs graduates of pedagogical and classical Universities. Respectively, hereafter the level of mathematical training of pupils and students will mostly depend on them, and their level of mathematical culture will advance the pupils' level of mathematical culture. But the experience of the last few years demonstrates that the level of mathematical culture of a significant number of the graduates of mathematical departments is rather low and the trend to its deteriorating is observed. Thus we observe a contradiction between the objectively required and highly demanded level of the mathematical culture of the mathematics departments' graduates and a real state of affairs. Such contradiction specifies the topicality of the issue under consideration.

The aim of this paper is to present the author's point of view as to the role of classical and pedagogical Universities in the development of students' mathematical culture.

The idea of a University is one of the most powerful ideas of the European cultural tradition. European culture is impossible without Universities which used to be and still are the centers of saving, formation, and increase of scientific, educational and cultural potential. The principle of universality (Latin: *universitas*, "*a whole*", "*a corporation*") is laid in the basis of the idea of a University. The importance of this principle concerns both a University as a scientific, educational and cultural institution and graduates that are the subject and objective of the University activity. Classical and pedagogical universities, accumulating knowledge and cultural gains of mankind, are directed at the training of intellectual specialists who would have universal education, high professional and universal culture. It seems reasonable that now, during the global transformations and crises (the most dangerous crisis in modern culture is that of a human being, which is connected with the loss not only of the possibility to comprehend the values of good, truth, and beauty but of elementary self-awareness as well) the issue concerning the future and prospects of classical and pedagogical universities becomes very important, because it is the universities that define the level of educational and cultural standards in the society.

For a long period of time tertiary education has been recognized mainly as the way to reach some high social status and a diploma was an evidence of belonging to mid-class. But the development of the society and the accessibility of higher education which took place under the slogan "democratization of the access to higher education" in the 60s-70s of the XXth century in Europe and in the 90s in the post-Soviet countries, generated the increase of the number of students more than twofold in comparison to the Soviet period and, at the same time, caused reasonable doubts concerning the status of tertiary education.

Confidence in higher education in Ukraine and respect to it mostly depends on the prestige gained in the Soviet period. We can observe the process of the devaluation of education because more than three hundreds of higher education institutions award identical state standard diplomas. The society has lost value orientations and a lot of people can't understand the essence of the notion "genuine higher education". It should be noted that the difference between the educational levels of the classical universities graduates and other higher education institutions graduates has almost vanished.

During the last two decades both in Ukraine and in other countries we can observe the overproduction of personnel, and as a result, they aren't demanded either in manufacturing or in social institutions. The training in higher education institutions has

---

[12]Mykhalin, G. O. (2003). *Professional training of the mathematics teacher while teaching mathematics analysis*. Kyiv: RNNTs "DINIT". (in Ukr.).

[13]Rozanova, S.A. (2003). *Mathematical culture of technical university students*. M.: FIZMATLIT. (In Rus.).

[14]Kuzminskyy, A. I., Tarasenkova, N. A., & Akulenko, I. A. (2009). *Scholarly premises for the methodical preparation of the future teachers of mathematics*. Cherkasy: Publishing house of Bogdan Khmelnitsky National University at Cherkasy. (in Ukr.).

an intermediate character between "knowledge accumulation", "prevention unemployment", and an artificial prolongation of the beginning of an adult life. The mass enrollment of the young people to the universities becomes less and less motivated and it is equated with the possibility to taste the peculiar life – the life in the student environment without obligations and responsibility.

Also, there are deeper reasons which are caused by the anxiety about the role of classical and pedagogical universities. The reproduction of some culture and its continuance always have a form of social inheritance, which means the transfer of the socially significant experience from the older generations to the younger ones. In the context of a separate individual, the process of social inheritance is a form of socialization, which means the mastery of some external general social experience and its transformation into the personal form of the existing culture. It should be noted that the volume of modern culture is huge and it is continuously increasing. At the same time the possibilities of an individual have some constraints and limits. An individual has to acquire some cultural information peace by peace. This factor is important for the explanation of the duration and heterogeneity of the socialization process. The greater array of information requires longer and more difficult process of mastering. The analysis of cultural traditions of a lot of civil societies shows that for more ancient cultures the value of traditions is of greater importance, and a greater part of social experience is passed in a non-formalized way[15] (Petrov, 2004).

The peculiarity of the European tradition is the using of the formalized way for the transfer of the social experience. Schools and Universities are the most important social institutions involved into this process. In the 60s-70s of the XXth century, schools and Universities shared the social experience of such volume and completeness that it was enough to support the saving and development of the European culture. Later the situation changed because the crucial increasing of the scientific and other knowledge and their differentiation formed a unilateral and very narrow specialization which was transformed into a "mutual isolation of disciplines".

Now it is widely known that no modern expert can account on a genuine interest to their own professional domain from the experts focused on the other domains both at the university and outside. It is a so-called "*situation de passage* ", when the experts from different domains of the same educational institution during the informal conversation can communicate in two different ways. In the first case serious scientific problems can be chosen as a subject of the conversation, but the discourse is held rather on a "low" level (by using the thesaurus of a secondary school); in the second case the subject of the conversation doesn't concern the scientific problems ("weather talk"). So, now we can't consider the university education to be a versatile one, and it means that the universities are losing the leading role in saving and developing the European cultural traditions. We can observe the "failure" in the system of the support of the European type of culture because it is becoming the culture of narrow experts with significant problems in professional mutual understanding. A lot of cultural experts, philosophers, theoreticians, and practitioners of higher education demonstrate the concern and anxiety about such situation.[2]

As for mathematical literacy and mathematical culture the following ideas can be formulated: 1) the volume of mathematical knowledge increases as rapidly as in the other sciences, but separate parts of accumulated mathematical knowledge aren't rejected for the reasons of being outdated and obsolete, which sometimes takes place in other sciences (like the Caloric theory in physics). As a rule, we observe the reconsidering and improving of knowledge; 2) "*situation de passage*" also happens, but mathematicians, as a rule, can reach mutual understanding; 3) nonetheless the unity of

---

[15]Petrov, M. K. (2004). *History of the European cultural tradition and its issues*. M.: ROSSPEN. (In Rus.).

mathematics is maintained. We observe a wider interpretation of the subject of mathematics and its more diverse and precise methods; 4) the mentioned above unity of mathematics is reached due to development of more higher levels of abstraction and the creation of common elements (at first set theory, then category theory and so on), which forms the basis of mathematics itself; 5) the role of classical and pedagogical Universities as main "producers" of mathematical culture and personnel increases.

Among the main reasons of the increase of the role of classical and pedagogical Universities the following ones can be identified:

1. Now, we can observe the mathematization of the world that includes the design of mathematical models of physical, chemical, biological, economic, social, and other phenomena. There can also be observed the use of mathematical concepts and samples to describe some properties of Nature and natural processes, for example - in designing machines and mechanisms they make use of mathematical models, in constructing buildings they use geometrical principles, and making calculations is compulsory in architecture etc. We can observe a great contribution of mathematics into the transformation of the whole world and in the formation of the well-structured and ordered mathematical Universe. The application of mathematical modeling in other sciences can be considered as the implementation of mathematical approaches in the intellectual domain[16] (Simakov, 2008). Mathematization of the world requires the availability of highly skilled specialists. Personnel for the scientific, educational, financial, economic, and other organizations are graduates of the classical and pedagogical Universities.

2. The decrease of the proportion of the students of mathematical faculties with respect to the total number of students. From 1991 to 2013 the number of students of higher educational institutions of the III-IV levels of accreditation increased from 0,88 mil. to 1,824 mil., at the same time the number of students of mathematical faculties stays almost the same. It means that the proportion of mathematics students in the total number of students has decreased twofold. It means that the role of students who graduate from the mathematical faculties of classical and pedagogical Universities is becoming more important.

3. The reduction of the interest of young people to difficult sciences and to mathematics in particular. It is a generally recognized fact for Ukraine and for other countries. A striking example of such situation is the decrease in the number of school leavers who want to get mathematical education. With exception of a few top Universities (in recent years they also have been experiencing problems), the rest ones are content with the enrollment of the students whose mathematical training leaves much to be desired. Between 2013 and 2016 the chronic shortage of the applicants and first-year students became the common problem for many Universities. Under such conditions it is very difficult to predict the increase of the number of students of mathematical faculties even in case of the simplification of the admission rules. It is also the argument in favor of the increase of the role and value of the graduates of mathematical department for the society.

4. Only classical and pedagogical Universities are able to train the personnel that are highly skilled in mathematics and also have a high mathematical culture (it ought to concern not only the doctors of science and PhDs, but all graduates). Now there is the movement in support of the professionally directed mathematical education for the student from non-mathematical faculties (of course, if mathematics is on their curricula). With no doubts in the idea of professionally directed mathematical education, we have to note that mathematics as a very special subject that has specific methodology, and as a unique phenomenon of the human culture has disappeared from

[16]Simakov, M. Yu. (2008). *Mathematics and the world.* M.: Samoobrazovanie. (In Rus.).

sight of lecturers and students. It means that the probability to meet an expert possessing a high level of mathematical culture among the graduates of non-mathematical faculties is rather small.

5. Despite the significant outflow of highly skilled experts and gifted young people, classical and pedagogical Universities still have the faculty that can save and develop high mathematical culture and traditions.

A secondary school mathematics teacher is one of the most popular professions which is chosen by the graduates of mathematical departments of classical and pedagogical Universities. That is, the teacher of mathematics is responsible for the formation of mathematical culture of society. No doubt, that a mathematics teacher can successfully carry out such difficult and important mission only if they themselves are the carriers of a high mathematical culture.

Reflecting on mathematical education and mathematical culture of the teacher of mathematics, the famous mathematician and pedagogue B. V. Gnedenko[17] (2006) maintains that "Undoubtedly, they must be able to solve the competitive tasks and explain methods of their solution on the sufficient level. They must be trained in mathematical analysis, analytical and differential geometry, algebra and probability theory. It is also useful to know the basis of functional analysis and theory of the function of complex variable, and know the ties of this theory with modern physics and aerodynamics. All these subjects are the foundation of mathematical knowledge of an educator.

Next, we are to speak on the history and methodology of mathematics. Without the history of mathematics, a teacher may find oneself in a problem situation because of some gaps in the knowledge concerning the paths of the development of mathematics, the main mathematical notions, and the names and heritage of famous scientists. The teacher will not be able to handle a very useful tool of the development of the pupils' interest to mathematics – historical facts; the teacher will not know about the formation of mathematical symbols, which are important for understanding the mathematics notations and their applications. Moreover it is important to be able to handle the programming, to know the basis of differential equations, operation analysis and principles of mathematical statistics. All topics mentioned above form the foundation of the mathematical education of a secondary school teacher of mathematics. Possibly, the list of a very wide complex of mathematical subjects necessary to secondary school mathematics teachers may seem odd based on the assumption that school teachers must teach only elementary mathematics, the beginnings of mathematical analysis and, probably, elements of programming. But, a teacher must know more than their pupils must know after leaving schools. Teachers must be familiar with a wide range of mathematical subjects in order to develop the interpretations from the position of modern mathematical culture, be ready to give consultations to the colleagues who specialize in physics, biology, and sociology" [18] (Gnedenko, 2006).

The main objective of classical and pedagogical Universities is to train such teachers. It seems reasonable that classical and pedagogical Universities train the personnel for the higher educational institutions of the III-IV levels of accreditation and the demands to the level of their mathematical literacy and mathematical culture is higher than to the level of a secondary school teacher. At the same time, training of the personnel for the national economy and education isn't the only possibility for the classical and pedagogical Universities to influence the formation and level of the mathematical culture of society. Such influence is broader and deeper. Almost all scientific,

---

[17]Gnedenko, B. V. (2006). *On teaching mathematics at universities and pedagogical institutions at the turn of the trird millenium.* M.: ComKniga. (In Rus.).

[18]Gnedenko, B. V. (2006). *On teaching mathematics at universities and pedagogical institutions at the turn of the trird millenium.* M.: ComKniga. (In Rus.).

methodical and popular science books for the secondary schools and higher educational institutions are written by the lecturers of classical and pedagogical Universities. They also support the process of the retraining mathematical education of the educators. In professional mathematical and pedagogical University environment the conceptions of the future reformation and development of the Ukrainian education are formed. University lecturers work with the gifted young people from secondary schools in the system of Minor Academy of Sciences of Ukraine (MAS), mathematical circles and during the mathematical competitions. The contribution of the members of classical and pedagogical Universities in the development of mathematics and modern approaches to learning of mathematics is cardinal. Moreover, spreading the thoughts and ideas of professional mathematical and educational communities in media, organizing public discussions and live communication are the important factors of the formation of mathematical culture.

Understanding of the importance of saving and development of mathematical and cultural mission of classical and pedagogical universities in the modern dynamic Ukrainian society leads to the following conclusions, which are possibly controversial.

1. Drastic concentration of the state and the society on the science and scientific and technical advance that are impossible without mathematics and mathematical education is required. Such concentration can provide the constant increasing of the operational efficiency and social standards[19] (Arnold, 1997). Without such turn the existence of Ukraine is disputed and the opinions concerning the mathematical and cultural missions of Ukrainian classical and pedagogical Universities become pointless. Next conclusions have been formulated with the assumption that described turn is possible.

2. Decreasing of the mathematical culture of the society and further aggravation of the negative consequences seems to be unavoidable without drastic enhancing of the public prestige and material welfare of the mathematicians, lecturers of higher educational institutions, and teachers of secondary schools.

3. Increasing of the general recognition of the work of the mathematicians and pedagogues makes it possible to increase the demands to scientific qualification, professional competence and professional results.

4. Increasing of the prestige of the occupations demanding fundamental mathematical training will facilitate the increase of the number of students who would like to be enrolled to the mathematical faculties and respectively will raise the level of requirements to the future students.

5. Increasing the requirements to pupils, students, teachers, and lecturers concerning their mathematical competence and culture must be gradual and in accordance with the programs and examples of tasks published in advance (not less than a year back), which describe the level of the demands and the impartial assessment procedure. In our opinion it should be the External independent testing (EIT).

6. Now, EIT is compulsory for the graduates of secondary schools and according to the obtained results they can be enrolled to the University. Furthermore the confirmation of the bachelor's degree should include the compulsory EIT in mathematics as the pre-condition of admission to Master course as well as the verification of master's degree must also include EIT as the pre-condition of admission to a post-graduate course.

7. Secondary school mathematics teachers are to take the EIT with pupils. The assessment can be treated as successful in case when the number of correct answers is more than two thirds of the maximal score.

[19] Arnold, V. I. (1997). Mathematics and mathematical education in the modern world. *Mathematical Education*, *2*, 22–23. (In Rus.).

8. Mathematics lecturers of higher educational institutions must pass EIT with bachelor students. The assessment can be treated as successful when the number of correct answers is more than two thirds of the maximal score.

9. Regular organization of EIT in mathematics for the pupils, students, teachers and lecturers shows the ways of its further improvement and use for the improvement of the regular mathematical self-education and mathematical culture.

10. Practice of the permanent revision and improvement of mathematical curricula both for secondary schools and higher educational institutions does not seem reasonable. Moreover it can cause the worsening and the disbalance of such curricula and the loss of the interdisciplinary connections. Undoubtedly current programs need to be updated by changing content and methodological approaches accordingly to the demands of the modern world. But such updates should be carried out very responsibly and after careful consideration, and no often than after 5-10 years with the exception of few modern Master courses. Stability of programs makes it possible for the authors to prepare a greater number of textbooks of high quality and organization of the methodical support; besides, teachers will have time to master these textbooks.

11. The situation when Ministry of Education and Science of Ukraine disregards the translation into the Ukrainian language and the publication of the best foreign textbooks on different mathematical topics (some kinds of mathematical bestsellers), popular science literature and books for children causes the lack of understanding. The formation of the effective system of translation into Ukrainian and the publication of mathematical literature mentioned above could be useful for increasing the level of mathematical culture of different target groups and assisting in the popularization of the Ukrainian language.

12. Popularization of mathematics as science and as occupation on TV, radio, in mass media and Internet is very noble and important for our state. We hope that the mutual efforts of state institutions and enthusiastic, gifted and obsessed with mathematics scientists, teachers, and popularizers of science will revive the society interest to the mathematics and its boundless potentials, particularly among the youth.

CHAPTER TWO

MATHEMATICAL PROBLEMS AND THEIR SOLVING

2.1. Methodology for Mastering Methods of Solving Mathematical Problems*

D. Millousheva-Boikina & V. Milloushev

*The article is published in the author's translation

Introduction. The main purpose of the school training is to prepare students for their future realization in life. The the education in mathematics and particularly, training in problem solving, plays a significant role for the achievement of this goal, since it is an essential instrument for the development of thinking of each individual. For the efficiency of this training, it is essential that its organization is based on modern principles and approaches such as the activity approach, the reflexive approach and the reflexive-synergistic approach. The development of pedagogical reflection (see Vasilev[1] (2006), Milloushev[2] (2009a), etc.) is a global modern trend, which is aimed to providing opportunities for self-development, self-control, self-regulation, self-organization and a serious motivation of students and turning students into active subjects in the learning process. According to S. Grozdev[3] (2007) „the rapid development of time in which we live requires dynamic changes of goals and tasks of the education of learners, who have to master certain volume of knowledge as well as highly effective methods for independent thinking, preparation and action. One of the main roles of education ... is gradual and permanent transformation of students from trained into self-learning".

An overview of recent publications on the topic. The problem about *training through math problems* has been developed by a number of prominent foreign and Bulgarian experts in methodology of education in mathematics, such as: D. Poya[4] (1972), Y. M. Kolyagin & V. A. Oganesyan[5] (1980), L. M. Fridman & E. N. Turetskii[6] (1984), V. G. Boltyanskii & Y. I. Grudenov[7] (1988), A. B. Vasilevskii[8] (1988), I. F. Sharigin[9] (1989), V. Y. Krupich[10] (1995), N. A. Tarasenkova[11] (2013), E. Skafa & V. Milloushev[12] (2009), K. Slavov[13] (1969), P. Petrov &

---

[1]Vasilev, V. (2006). *Reflection in knowledge, self-knowledge and practice*. Plovdiv, Bulgaria: Makros. [In Bul.]

[2]Milloushev, V. B. (2009a). Reflection and reflective approash in mathematics education. – In journal: *Vestnik Cherkassky University of Bohdan Khmelnytsky, Seria Pedagogical Science*, (143), (pp. 56-69). Cherkasi (Ukraine). [In Rus.]

[3]Grozdev, S. (2007). *For high achievements in mathematics.The Bulgarian Experience* (*Theory and Practice*). Sofia, Bulgaria: Kota. (p. 62).

[4]Poya, D. (1972). *How to solve a problem*. Sofia, Bulgaria: Narodna prosveta. [In Bul.]

[5]Kolyagin, Y. M. & Oganesyan, V. A. (1980). *Learn to solve problems*. Moskva, Russia: Prosveshtenie. [In Rus.]

[6]Fridman, L. M. & Turetskii, E. N. (1984). *How to learn to solve problems*. Moskva, Russia: Prosveshtenie. [In Rus.]

[7]Boltyanskii, V.G. & Grudenov, Y.I. (1988). How to learn to look for solutions of problems. – *Mathematics in school,* (1), (pp. 8-14). [In Rus.]

[8]Vasilevskii, A.B. (1988). *Trainging in solving problems in mathematics*. Minsk, Belarus: Visheishaya shkola.

[9]Sharigin, I. F. (1989). We are learning to solve geometry problems. – *Mathematics in school,* (2), (pp. 87-101). [In Rus.]

[10]Krupich, V.Y. (1995). *Theoretical basis of the education in solving school mathematical problems.* Moskva, Russia: Prometei. [In Rus.]

[11]Tarasenkova, N.A. (2013). Features coding geometric concepts. – *Science and Education a New Dimension: Pedagogy and Psychology*, (5), (pp. 7-11). [In Rus.]

[12]Skafa, E. & Milloushev, V. (2009). *Constructing educational-cognitive heuristic activity in solving mathematical problems.* Plovdiv, Bulgaria: University press „Paisii Hilendarski". [In Bul.]

[13]Slavov, K. (1969). *Basic methods for solving problems in algebra*. Sofia, Bulgaria: Narodna prosveta. [In Bul.]

D. Millousheva-Boykina[14] (2000), I. Ganchev[15] (2001), M. Georgieva[16] (2001), R. Mavrova & D. Millousheva-Boykina[17] (2002), S. Grozdev[18] (2002), V.B. Milloushev & D.G. Frenkev[19] [20] (2004a); (2004b), I. Tonov[21] (2005), V. Milloushev[22] (2009b), etc. Some of them focus on learning generally-logical and private-mathematical methods for solving mathematical problems. This is most effectively accomplished through the selection and creation of suitable teaching math problems that are separated in appropriate didactic systems. The development of such systems of problems is based on appropriate theoretical formulations based on the concept of Vygotsky-Ganchev[23] (Ganchev & Kuchinov, 1996) about the zones of "actual" and "near" development of students (ZAD and ZND), as well as the ideas for using the so called problems-components in accordance with the reflexive-synergistic approach[24] (Milloushev, 20016), since there is a close connection between the relationships "education – reflection" and "education – development", and it is well known that the learning precedes the development.

The creation of didactic systems of mathematical problems especially for mastering private-mathematical methods having not only a practical meaning for solving certain problems but being also important in ideological terms, contributes to the formation and development of praxeological reflection in students. The most important ideologically private-mathematical methods are the method of auxiliary circle (the methodology of mastering is described in the article[25] (Boykina & Milloushev, 1995), the method of auxiliary angle (which, as the method of auxiliary circle appeared to be special cases of the method of "auxiliary" element), the method of parameterization[26] (Portev, 2001), the graphic method[27] (Milloushev & Boykina, 2002), the method of mathematical induction[28] (Milloushev, 2002) and others.

---

[14]Petrov, P. & Millousheva-Boykina, D. (2000). About the skills for solving and formularing of mathematics problems. In: *Scientific Papers of PU „Paisii Hilendarski"*, 37 (2), (pp. 17-23). [In Bul.]

[15]Ganchev, I. (2001). The analysis and synthesis in mathematics education. In: V. Miloushev (Eds.), *Methods for solving problems* (*from the school course in mathematics*). Part I. (pp. 177-193). Plovdiv, Bulgaria: Makros. [In Bul.]

[16]Georgieva, M. (2001). *Reflection in mathematics education (V-VI class).* V. Turnovo, Bulgaria: Feber. [In Bul.]

[17]Mavrova, R. & Millousheva-Boykina, D. (2002). Method of undetermined coefficients. In: V. Miloushev (Eds.), *Methods for solving problems* (*from the school course in algebra and analysis*). Part II. (pp. 37-46). Plovdiv, Bulgaria: University press „Paisii Hilendarski". [In Bul.]

[18]Grozdev, S. (2002). Organization and self-organization in solving problems. – *Mathematics and informatics*, (6), (pp. 51-58). [In Bul.]

[19]Milloushev, V.B. & Frenkev, D.G. (2004a). Models for solving and creating geometry problems for finding out properties. In: *Mathematical education: Status and perspectives*. (Proceedings of International Conference), (pp. 22-28), Mogiljov, Belarus: MDU, (A plenary lecture at invitation). [In Rus.]

[20]Milloushev, V.B. & Frenkev, D.G. (2004b). A model for joint implementation the methods of analysis, synthesis and parameterization in searching for solutions of geometric problems. In: *Mathematics and Mathematical Education,* (pp. 348-353). Sofia, Bulgaria: BAS. [In Bul.]

[21]Tonov, I. (2005). The education of theachers in mathematics through solving problems. – *Mathematics and informatics*, (3), (pp. 5-10). [In Bul.]

[22]Milloushev, V. (2009b). About classification of methods for solving mathemdtics problems. – *Scientific Papers of PU „Paisii Hilendarski"*, 46 (2), (pp. 77-90). [In Bul.]

[23]Ganchev, I. & Kuchinov,Y. (1996). *Organization and methodology of the lesson in mathematics*. Sofia, Bulgaria: Modul. [In. Bul]

[24]Milloushev, V. (2016). Reflective-synergetic approash in mathematics education – *Strategies for Policy in Science and Education*, Bulgarian Educational Journal, 24 (1), (pp. 69-85). [In Bul.]

[25]Boykina, D. V. & Milloushev, V. B. (1995). Geometry problemss that are solved by using an auxiliary circle. – *Scientific Papers of PU „Paisii Hilendarski"*, 32 (2), (pp. 145-152). [In Bul.]

[26]Portev, L. B. (2001) Definitness (parametrization) of geometrical figures and applications. In: V. Miloushev (Eds.), *Methods for solving problems* (*from the school course in mathematics*). Part I. (pp. 177-193). Plovdiv, Bulgaria: Makros. [In Bul.]

[27]Milloushev, V. & Boykina, D. (2002). Graphical method. In: V. Miloushev (Eds.), *Methods for solving problems (from the school course in algebra and analysis).* Part II. (pp. 47-65). Plovdiv, Bulgaria: University press „Paisii Hilendarski". [In Bul.]

[28]Milloushev, V. B. (2002). Method of mathematical induction. In: V. Miloushev (Eds.), *Methods for solving problems* (*from the school course in algebra and analysis*). Part II. (pp. 13-36). Plovdiv, Bulgaria University press „Paisii Hilendarski". [In Bul.]

Purpose of the research. One of the main purposes of this paper is to develop a theoretically-practical basis for constructing a methodical system for effective work with mathematical problems, so that the system to be compatible with modern didactic principles and approaches, as well as with current interpretations of a number of key categories in the field of methodology of mathematics education. In this regard, beside the important role of reflexive-synergistic approach, we would like to note the significance of the ideas and notions about the relations between education and development. In order to achieve a better accomplishment of the continuity in the development of this methodical system, we use the research of the contemporary leading methodologists mentioned above and especially our teachers Slavov[29] (1969) and Portev[30] (1996), whose research contained some elements of the theoretically-practical basis for developing the practical aspects of this methodical system.

In this paper we aim to present part of our experience, related to the teaching of learners (pupils and students – future teachers) in methods for solving problems from different sections of school mathematics. In this connection, we will give general ideas and specific implementation through examples for mastering the basic generally-logical methods and some private-mathematical methods for solving problems of school mathematics. For this purpose we will present some opportunities for construction of didactic systems of mathematical problems, designed to mastering specific methods and heuristics for solving problems.

Exposure of the basic material. The thesis of the research is the following: the mastering of methods approaches and heuristics that are applied not only to implementation but also in searching and discovering solutions of mathematical problems is essential to the achievement of the aims of mathematical education. Our practical pedagogical experience shows that the mastering of knowledge of a particular theme from the school course in mathematics and skills to use this knowledge, on one hand, and mastering of knowledge about the nature of generally-logical and private-mathematical methods and skills for their application – on the other hand, can be carried out in accordance with the reflexive-synergistic approach with appropriate structuring of systems of mathematical problems relevant to the specific educational and developing purposes. Using such systems of problems the students can also master different heuristics for looking for a solution which, although not always leading to positive results, contribute to the achievement of purposess. The realization of all this can contribute to self-awareness in maximum degree of both the positive and the negative role of the different methods and heuristics in solving problems.

Thinking at atomic, molecular and cellular level plays a significant role in the activity of solving mathematical problems[31] (Ganchev, 1999). Part of the private-mathematical methods for solving, the so called standard problems (for example, the ones that are related to the solving of linear, square, biquadratic equations and etc., which may be considered as mathematical models in "canonical form") are based on certain formulas (theorems-properties). Specific algorithms for applying these methods have been developed. That’s why it is appropriate that the individual use of such methods be considered as an activity at atomic level.

The multiple joint applications of certain mathematical knowledge and elements from propositional logic (mostly properties related to implications of statements and inferences rules) is briefly described as an activity at molecular level. According to us, it is appropriate to consider the use of generally-logical methods, which are based on

[29]Slavov, K. (1969). *Basic methods for solving problems in algebra*. Sofia, Bulgaria: Narodna prosveta. [In Bul.]

[30]Portev, L., Mavrova, R., Milloushev, V., Nikolov, N., Kozhuharova, R., Bizova, G. & Makrelov, I. (1996). *Guidance on teaching practicum for students in mathematics at PU „Paisii Hilendarski”* (3-rd ed.). Plovdiv, Bulgaria. [In Bul.]

[31]Ganchev, I. (1999). *Basic learning activities in mathematics lessons* (*synthesis of results from different studies*). Sofia, Bulgaria: Modul-96. [In Bul.]

elements of propositional logic (for example, the method of contraposition and others) as an activity at molecular level as well.

Besides the possible implementation of certain parts of the solution by activities at atomic and molecular level, as an activity at cellular level we consider also the separating of the solving of a certain problem into individual parts - the so called problems-components, and their consecutive implementation in view of receiving the solution of the given problem. This action is usually performed on the basis of generally-logical methods synthesis, analysis and combinations of them, as well as on relevant reasoning skills at atomic and molecular level, including the use of appropriate private-mathematical methods for solving the separated problems-components.

From the foregoing, we destinguish the importance of mastering generally-logical and private-mathematical methods which play the role of basic "operational" instruments for carrying out the activity of solving mathematical problems. But in order to turn these methods into instruments, it is necessary to make them also purpose of education on a certain stage.

The question about effective utilization the possibilities of reflexive approach in mathematics education is studied by M. Georgieva in the monograph[32] (Georgieva, 2001), where a model of the structure of the system of the categories "perception", "remembering", "understanding", "reflection", "application" and "mastering" is presented (we quote this model on Fig. 1 for convenience of the reader).

Analyzing the connections on Fig. 1, it is clear that:

1) understanding and rationalization (as well as perception and remembering) can be implemented even without reflection;

2) reflection is achieved through understanding and rationalization;

3) reflection is in the base of application and mastering, as well as in the formation of reflexive abilities.

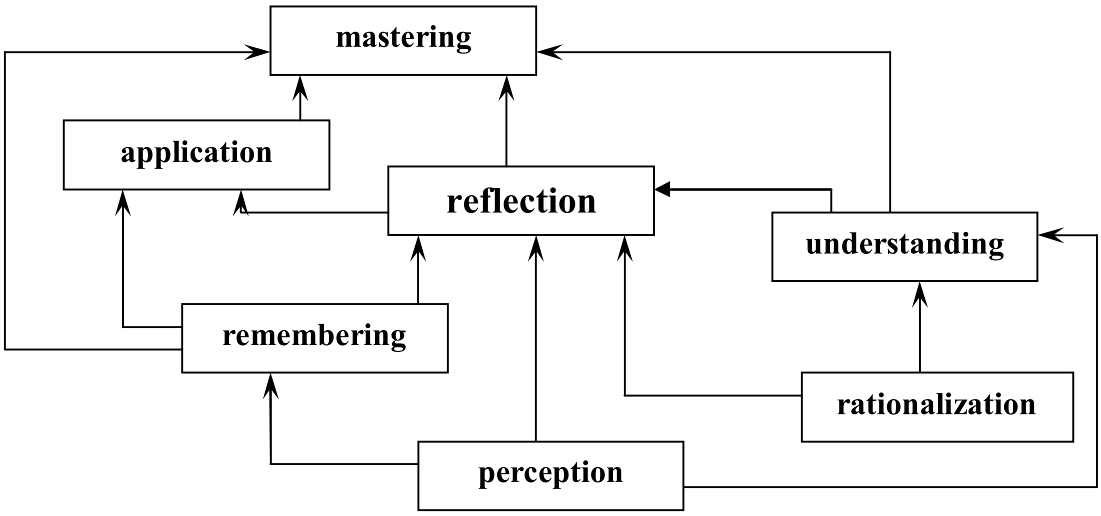


Figure 1. Schematic model of the place of reflection and consequence of mental processes prior mastering

From this ascertainment, one concludes that in the mathematics education the “internal” experience of the learner is important, because it is an essential condition for the development of reflection. The key point of mastering an experience in solving various problems is the assimilation of the ***essence***, because it leads to the formation of theoretical thinking and development the intellect of the learner. In this context,

[32]Georgieva, M. (2001). *Reflection in mathematics education* (*V-VI class*). V. Turnovo, Bulgaria: Feber. (p. 12), [In Bul.]

the priority is given to the relationship between the assimilation and the ***application*** of experience, in view of satisfying the learner's cognitive need. This confirms the connections between the intellectual and praxeological reflection, which we show through the complex model for mastering and applying mathematical knowledge and skills, presented in the article[33] (Milloushev & Frenkev, 2008). Reflection is important for purposeful utilization of "mathematical experience" during which process various activities requiring high intellectual tension are carried out. It leads to the need of effective use of the possibilities of active approach in mathematics education. In this connection the following question arises: "which side of the active approach can serve as a basis for a new strategy of mathematics education today having in mind the new reality?" A definite answer to this question is given in the monograph[34] (Georgieva & Grozdev, 2015), where a new dynamic modification in the educational system, called NDM-paradigm is offered. In the base of this paradigm stands the morfodinamics for the development of noosphere intelligence of personality for lifelong learning. Rethinking of NDM-paradigm leads to a new approach in the development of intellect, depending on the basic components of the cognitive structure of human creative activity. As the authors emphasize, "the model provides an opportunity to achieve the highest level of self-organization, self-knowledge, self-actualization, self-realization and self-evaluation that lead to the search of optimal development of intelligence within the lifelong learning"[35] (Georgieva & Grozdev, 2015).

If we consider the schematic model on Fig. 1 in terms not only of reflexive approach, but also of reflexive-synergistic approach, then we can reveal the place of self-organization in the system of cognitive processes prior to mastering, as well as the no less important process (on mega level) – future applications of the mastered knowledge in solving relevant problems. A similar model is presented on Fig. 2.

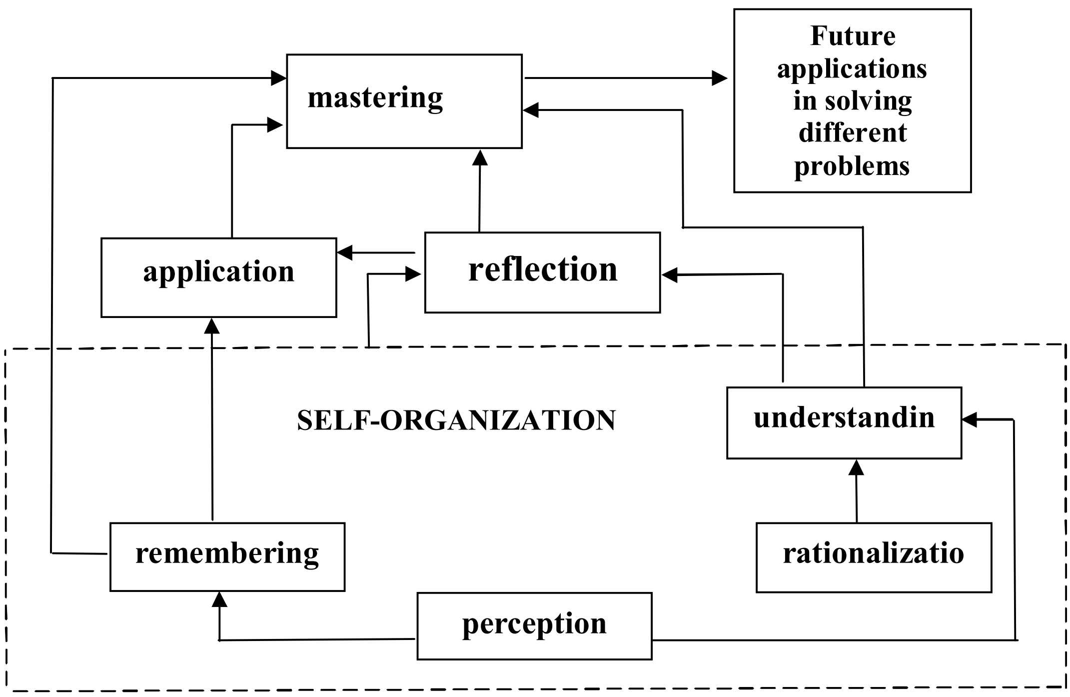


Figure 2. Schematic model of the place of reflection and self-organization and the sequence of cognitive processes prior mastering

[33]Milloushev, V. B. & Frenkev, D. G. (2008). About one reflexive model of education and its application. In: *Mathematics and Mathematical Education,* (pp. 61-72). Sofia, Bulgaria: BAS. [In Bul.]

[34]Georgieva, M. & Grozdev, S. (2015). *Morfodinamics about the development of noosphere intelligence*. Sofia, Bulgaria: Iztok-Zapad. [In Bul.]

[35]Georgieva, M. & Grozdev, S. (2015). *Morfodinamics about the development of noosphere intelligence*. Sofia, Bulgaria: Iztok-Zapad. (p. 302) [In Bul.]

We will note that the *new thing* in this model is based on the following:

1) the components perception, remembering, rationalization and understanding of the model on fig. 1 *are included in self-organization*, since on the certain stage of education the students can reach self-actualization and self-development;

2) mastering is not only preceded by the component applications, but also leads to *future applications* to solve problems of a different nature.

The ideas above should be reflected on the construction of systems of educational mathematical tasks designed to master certain methods for solving, so that the training to be oriented to summarized ways of activity through them. Moreover, the relationships between different systems (which differentiate them as components of a broader "global" system) *can ensure step by step implementation of the following phases of education*:

- *first phase*: **self-organization**, expressed in **perception - remembering – rationalization - understanding;**
- *second phase:* **reflection;**
- *third phase:* **applications – mastering – future applications.**

Georgieva (2001) identifies in her study[36] three types of relationships: "education – reflection", "education – development" and "education – development –reflection". She uses as a base the concept of N. F. Talyzina[37] about the process of learning in terms of the relationship "education – reflection" and pays attention to the following two "types of the approach to the learning of students in class:

♦ the teacher develops educational content and presents it to students, and they carry out relevant activities;

♦ students participate actively in the rediscovery of scientific truths and acquire their own cognitive experience"[38] (Georgieva, 2001).

Accepting the thesis that the place of reflection can be searched in both varieties – **over the knowledge** and also **over the activity**, we are going to present models for mastering private-mathematical methods through "rediscovery".

On the base of the close connection between the relations "education – reflection" and "education – development" and having in mind the perception, that "education is good when it goes before development"[39] (Vigotski, 1956), the principally-methodological significance of these concepts for the mathematics education can be defined. On this base, the accent is put on the importance of the concept "*mastering to the extent of random play and implementation the most general knowledge and skills which are not given directly in education*", introduced by I. Ganchev[40] (1999). We think that it is especially true for knowledge and skills related to the use of generally-logical methods. The further construction of the methodical system is considered with this fact.

The important means of acquiring knowledge and skills on reflexive level named by I. Ganchev[41] (1999) "*method of education by summarizing arguments*" also worths attention.

This method is based on the presumption that, having mastered a certain knowledge or skills, there must be aquired much more specific knowledge or skills that have the same characteristics as the first one, and also this knowledge needs to be "detached", separated from the concreteness of the knowledge and must be divided and fixed in the

---

[36]Georgieva, M. (2001). *Reflection in mathematics education* (*V-VI class*). V. Turnovo, Bulgaria: Feber. [In Bul.]

[37]Talizina, N.F. (1983). *Forming the cognitive activity of students.* Moskva, Russia: Znanie. [In Rus.]

[38]Georgieva, M. (2001). *Reflection in mathematics education* (*V-VI class*). V. Turnovo, Bulgaria: Feber. (pp. 16-17), [In Bul.]

[39]Vigotskii, L.S. (1956). *Selected psychological researches*. Moskva, Russia. (p. 449). [In Rus.]

[40]Ganchev, I. (1999). *Basic learning activities in mathematics lessons* (*synthesis of results from different studies*). Sofia, Bulgaria: Modul-96. (p. 47). [In Bul.]

[41]Ganchev, I. (1999). *Basic learning activities in mathematics lessons* (*synthesis of results from different studies*). Sofia, Bulgaria: Modul-96. (p. 60). [In Bul.]

mind of the subject. The performance of such a summarizing activity can be managed better and even accelerated. This can be achieved successfully using a more rational cooperation of students with adults (teachers). For this purpose, it's necessary teachers not only to select and offer students consistently different specific knowledge with one and the same structure, but they have to indicate clearly the community of this structure in definite moments. In this way they accelerate the "detachment" of the common properties or characteristics from their specific circulators, i.e. ensure the achievement of "their arbitrary mastering".

According to us the method of education by summarizing arguments contributes significantly to the development of cognitive (intellectual) reflection in learners as well. We will use the mentioned above and this fact to stack the three phases of "global" system for mastering certain generally-logical methods for solving problems mentioned above to the conclusions made by I. Ganchev in his monograph on the base of the model that he has constructed about the relationship between the education and mental development, that are also related to the question for the development of reflexive abilities of students in mathematics education.

In the *first phase*, generally said, there is carried out (under the guidance of the teacher) a *giving of knowledge and skills*, related to the application of a particular method (depending on the certain mathematical knowledge and skill for its implementation) in ZND – the zone of near development of the student; in the *second phase* a transition is made from ZND to ZAD – zone of actual development of the learner with the increased use of the reflexive approach; in a *third phase* - development of *reflexive abilities in ZAD*, which helps to achieve "mastering to the extent of random play and implementation the most general knowledge and skills" upon the generally-logical methods for solving problems, which according to the curriculum for the secondary school "are not given directly in education" indeed.

With a view to optimal formation of intellectual and praxeological reflection, the methodology for developing the skills of students in mathematics education must meet certain regularitites and requirements, some of which are contained in the research of I. Ganchev[42] (1999) and P. Petrov[43] (2003) and especially in their models about the relationship between education and mental development. One of these requirements is that learners must be taught to use the Papp scheme for solving problems without being told its name. According to the author, the ability to reason on this scheme "for the majority of students is still "attached" to the various specific types of problems, i.e. it is not mastered to a degree of random. In order to achieve the last one, it is necessary to gain experience on different specific types of cases, which experience have to rise on the level of mastering to the extent of random" [44] (Ganchev, 1999) under the guidance of adults (teachers) by summarizing arguments on a certain stage.

We believe that it's appropriate that the requirement given above is good to be applied also for mastering knowledge and skills for *combined* application of generally-logical methods (in different versions – see Milloushev[45] (2008)) for searching and finding out solutions. Having in mind that in the mathematics education in the secondary school there is carried out the so called faraway propaedeutic about the generally-logical methods analysis, synthesis and some combinations of them, then their

[42]Ganchev, I. (1999). *Basic learning activities in mathematics lessons* (*synthesis of results from different studies*). Sofia, Bulgaria: Modul-96. (pp. 145-147). [In Bul.]

[43]Petrov, P. D. (2003) *Forming skills for solving problems of the school course in mathematics*. Stara Zagora, Bulgaria: Kota. (p.71-76). [In Bul.]

[44]Ganchev, I. (1999). *Basic learning activities in mathematics lessons* (*synthesis of results from different studies*). Sofia, Bulgaria: Modul-96. (p. 145). [In Bul.]

[45]Milloushev, V. B. (2008). The t*riad of activities solving, creating and transforming mathematical problems in the context of the reflexive - synergetic approach.* Abstract of the dissertation to obtain the degree "Doctor of pedagogical sciences" in scientific speciality "Methodology of education in mathematics". Sofia, Bulgaria. (p. 35). [In Bul.]

further mastering can be implemented efficiently if it considers with those conclusions of the research of I. Ganchev[46] (1999), which are related to the methodology of organization of education and are directly related to the reflection. In particular, these conclusions should be reflected in the construction of a ***"global" system of teaching mathematical problems*** for mastering certain generally-logical and/or private-mathematical methods for solving problems in such a way, that it contains sub-systems with problems, aimed to achieve the following **objectives**:

a) establishing ZND of learners on generally-logical and/or private methods by test with relevant criterial problems, which must be accompanied by directions (in written or oral form, with a suitable organization of work);

b) for the learners, whose knowledge and skills about the methods are outside the ZND, there should be organized a collectively solving or solving with help and summarizing arguments of appropriate problems from Z_{ZND} with a view to ZND to be extended in the right direction;

c) establishing ZAD of learners through a test, containing questions or problems from different levels of maximum Z_{ZAD} for the certain group of learners;

d) for the subgroup of learners, whose knowledge and skills about the methods are outside the ZAD, there must be provided the following:

- make easier the insert of knowledge and skills for the certain method from Z_{ZND} in Z_{ZAD} using the method of education by summarizing arguments and expand Z_{ZAD} in the right direction to this specific knowledge of Z_{ZND};

- insert knowledge and skills about the certain method from Z_{ZND} in Z_{ZAD} through sustained independent work with close to them knowledge and skills from Z_{ZAD} or work with help, using the method of education by summarizing arguments. The work with help usually is a collective work with examples, giving directions, etc.;

e) consolidating knowledge and skills about the relevant method in Z_{ZAD} by appropriate independent work (in class or at home) with these or very close to them knowledge or skills from Z_{ZAD};

f) maintaining the achieved level of development in terms of knowledge and skills for the application of the method (i.e. non-disappearance of the respective higher mental functions from ZAD) by periodically setting an independent work (with special exercises for the purpose) with activities with relevant knowledge and skills from Z_{ZAD}.

We will emphasize again that "the method of education by summarizing arguments" is a key tool here, that is important and at the same time reveals the role and the place of reflection in the activities on the realization of those objectives.

When constructing the "global" system (as a means to achieve the aims of the relevant methodological system) the fact that it must be composed of subsystems which correspond to the objectives **a) – f)** should be considered.

The attitude "education – reflection" has a connection with the fourth requirement in the methodology for developing the skills of students to solve mathematical problems, formulated in (Ganchev[47], 1999), namely – creating didactic systems of indications and their use in the education under the guidance of the teacher (i.e. to work with them as a knowledge and skill from Z_{ZND}). For this purpose it is useful to record the systems of indications in a separate notebook and periodically recall and expand them with newly studied indications. This gives an idea when studying different ways for solving problems of one and the same section (for example, "Systems of equations of the second degree with two unknowns", "Trigonometric equations" and others) to create a "list" of the methods used to "attack" the problems of the studied section, and after acquainting

[46]Ganchev, I. (1999). *Basic learning activities in mathematics lessons* (*synthesis of results from different studies*). Sofia, Bulgaria: Modul-96. (pp. 64-65). [In Bul.]

[47]Ganchev, I. (1999). *Basic learning activities in mathematics lessons* (*synthesis of results from different studies*). Sofia, Bulgaria: Modul-96. (pp. 147-154). [In Bul.]

with each new method, to add it to this "list". We also consider it's appropriate in the process of acquainting students with a new kind of problems, connected with newly studied material, which are solved by using an already known method, first to update knowledge about this method and the skills for its application with several appropriately selected problems from already studied material, but of a different type, while focusing on its general applicability and comprehensiveness. Then to describe other types of problems, discussed so far, in the solving of which the certain method is successfully applied. As a result of this activity a "list" of the types of problems should be created (if so far is not made any), the "new" application of the threated method should be considered and the "list" with the type of problems to which it is applicable should be completed.

On the base of the four basic principles of reflective approach: *for activity, for consciousness, for reflectivity and for tolerance* presented in the book[48] (Vassilev, Dimova, Kolarova-Kancheva, 2005), we have created a ***structural model*** of the system "trained – trainer" in the context of reflexive approach in appropriate educational environment described in the article[49] (Milloushev, 2009). Considering the situation that the reflective approach, as a humanistic approach, no matter how it is described, emphasizes on the reflexive potential of the subjects in education (both the trainer and the trained). One of the aims of the reflexive approach is to establish the subject as self-updating and self-regulating personality. This approach refers to the expression of four valuable personal qualities of subjects in education, corresponding to these four basic principles.

Since the principle of *activity* requires from subjects primarily a performance of inner activity, then to realize interaction between the activity of the trainer and activity of the student aimed at realization of this principle, the trained must complete relevant specific tasks. Some of them, which are designed to master a variety of methods for solving problems, can be adapted, for example, as follows:

▪ for each generally-logical method must be constructed systems of meaningful mathematical problems, taken from diverse educational content which to provoke activity of students at different levels;

▪ organizing the academic activity in such a way, so that to generate positive reasons for learning, to stimulate the interest of the learner both to the learning material and to the methods used to solve problems from it.

In order to realize the principle of *consciousness* successfully, it is appropriate the trainer to put specific didactic tasks, adapted to mastering specific methods for solving mathematical problems, with a view to achieve interaction between the activity of the trainer and activity of the student towards the realization of this principle.

For example:

- setting such aims of the different subsystems of the "global" system designed to mastering the methods, which to be adequate to ZAD and ZND of the learner;
- selection of educational problem-components of the system, as well as educational actions, which to motivate the learner to carry out a certain educational activity;
- formation of skills of students to select independently appropriate methods for solving problems.

The principle of *reflexivity* requires greater activity, awareness and support of learners to understand what they do. Especially in the process of teaching methods to solve problems, the teacher not only motivate the student in the right degree (mostly by

[48]Vasilev, V., Dimova, J. & Kolarova-Kancheva, T. (2005). *Reflection and learning. Part 1: Reflection - theory and practice*. Plovdiv, Bulgaria: Makros. (p. 81-87). [In Bul.]

[49]Milloushev, V. B. (2009). Reflection and reflective approash in mathematics education. – *In journal: Vestnik Cherkassky University of Bohdan Khmelnytsky, Seria Pedagogical Science*, (143), (pp. 65-66). Cherkasy, Ukraine, [In Rus.]

focusing on the general applicability of the methods and on the fact that they are an effective means of dealing with multiple problems), but he directs him to self-organization of his own activity (self-actualization, self-realization, self-control, self-regulation, self-improvement of his personal qualities). For this purpose, the trainer should unfold the reflexive potential of curricula, which presumably includes various methods for solving mathematical problems and a theoretical basis for their separation and fixing in the mind on the subject. Along with this, it is necessary to create conditions for activating intellectual and/or praxeological reflection in the learner over the knowledge of appropriate methods and over their skills for their implementation. Some authors unite together the principles that we have mentioned by now in a single general *principles of activity, consciousness and reflexivity*. This is explained by the fact that the first one determines the second and both together – the third, i.e. they are hierarchically linked.

If adapted to the present study some of the conclusions in monograf we can say that in the education in methods and heuristics for solving problems, the activity is leading in most cases, that is why reflection is associated with the ability, with the skill, with the "active" mental new creation and then it is appropriate to use the term *reflection over the actions*. In some cases, however, knowledge is leading and then "reflection is seen as a process that translates the information into personal knowledge and in these cases it is about *reflection over knowledge*"[50] (Georgieva, 2001).

The elaborated in the same work conceptual model of technology for reflexive learning has a scientific and practical interest. Its structure includes: motives and objectives of reflexive activity, methods, didactic means and forms of organization of education and the separated four levels – **reproductive, productive, transfer and creative** – for acquiring knowledge and for carring out activities through which corresponding reflexive skills are formed.

From the point of view of the problem, threated in the present investigation, we will note that on the **reproductive level** the following actions are performed many times: monitoring, orientation (comparison – juxtaposition and confrontation, and corresponding conclusions) sample actions, acquaintance with the essence of a method or a heuristic approach to searching a solution. In order to ensure reflexive actions, in this way "there are created optimal conditions for the realization of the mechanism for taking decisions, respectively for formation of skills for self-regulation"[51] (Georgieva, 2001).

On **productive level** some phenomenological characteristics of reflection appear, such as: choice of cognitive schema or creation a new one in accordance with the experience of the learner, ability this cognitive scheme "to be objectified in different codes" while preserving "its invariant structure".

On the **transfer level**, on the base of the knowledge and skills acquired in the first and second level, the learner demonstrates the acquired experience, thus appears a necessity of reflection, i.e. at this level the learner subject applies combination of the most appropriate cognitive schemes.

On the **creative level** the same activities (as in the third level) are carried out, but now with new, unusual conditions which require of the learner generalized, transfer skills to carry out productive heuristic activity, associated with a certain amount of "ambiguity that allows various possibilities for realization of the reflexive experience"[52] (Georgieva, 2001), i.e. the creative activity in unfamiliar situations is fully realized.

---

[50] Georgieva, M. (2001). *Reflection in mathematics education* (*V-VI class*). V. Turnovo, Bulgaria: Feber. (p. 26). [In Bul.]

[51] Georgieva, M. (2001). *Reflection in mathematics education* (*V-VI class*). V. Turnovo, Bulgaria: Feber. (p. 33). [In Bul.]

[52] Georgieva, M. (2001). *Reflection in mathematics education* (*V-VI class*). V. Turnovo, Bulgaria: Feber. (p. 33). [In Bul.]

The system of these levels can be used as a model for construction (selecting, creating and/or transforming problems) of systems of educational mathematical problems, the purpose of which is also related to the mastering certain generally-logical or particularly-mathematical methods for solving problems.

The effective realization of one theoretically-applied research requires the development of adequate methodological tools for stimulating and developing the reflexive and synergistic thinking of students. That's why we use the separated by M. Georgieva[53] (2001) three levels of reflection, as they can serve as a model for education in accordance with the reflexive-synergetic approach as well. Indeed, any system of problems designed for mastering a specific type of methods and/or heuristics for solving mathematical problems, consists of three subsystems that correspond to the three levels of reflection:

First level. Learner subject orientates in a single performing of synthesis, respectively analysis or repeatedly committing only analysis or only synthesis when working with elementary problems of a specific topic from the school course in mathematics, correctly describes the mechanism of application of certain definitions and theorems, but still he is not able to explain properly the links between different steps of analysis or synthesis, on the base of the implicit use of the rule "hypothetical syllogism."

Second level. Learner interprets the nature and the appropriateness of the combined application of analysis and synthesis in different versions. While on the first level he processes the information in the problem on the base of certain definitions or theorems in their capacity of deductive rules, as his thoughtas "move" one way, most offend from the given to the sought, then on the second level he repeatedly two-way (from the given to sought and from sought to given) processes the information, arranging it according to a scheme of application of analysis and synthesis that is already mastered. In order to reach this level, the creation by the students the so called inverse problems plays an important role, as well as accomplishing various forms of transformation of mathematical problems, which contributes to the realization of the triad[54] (Milloushev, 2008) of activities: solving, creating and transforming problems. Thus the trained becomes more confident, assess his preparedness to solve the problems, his actions are aimed at intensive acquiring of knowledge, purposefully shows self-communion and self-evaluation of the activity.

Third level. Learner subject uses freely the mastered theoretical knowledge and methods for appropriate use in handling problems, transfer them into new situations, combine them with other knowledge and skills, as well as with certain heuristics. Now he analyzes, remodels, evaluates the information in the problem according to established criteria for evaluations and prognosis, often alone selects or produces the necessary information. His reflection is revealed by the certain criterion that has found a stable, unchanging place in his personality. Learner shows "readiness for self-development and self-improvement; conscious self-control and sound mind; developed ability to adapt to new conditions"[55] (Georgieva, 2001).

These levels of reflection can be transferred and adapted also in the construction of systems of problems for mastering, in the context of reflexive-synergistic approach, basic and essential generally-logical methods and particularly-mathematical methods and some more commonly used heuristics from teaching practice. It will be shown next

[53]Georgieva, M. (2001). *Reflection in mathematics education* (*V-VI class*). V. Turnovo, Bulgaria: Feber. [In Bul.]

[54]Milloushev, V. B. (2008). The t*riad of activities solving, creating and transforming mathematical problems in the context of the reflexive - synergetic approach.* Abstract of the dissertation to obtain the degree "Doctor of pedagogical sciences" in scientific speciality "Methodology of education in mathematics". Sofia, Bulgaria. [In Bul.]

[55]Georgieva, M. (2001). *Reflection in mathematics education* (*V-VI class*). V. Turnovo, Bulgaria: Feber. (p. 65). [In Bul.]

in this study, as first we will consider the question about assimilation the main generally-logical methods for solving mathematical problems.

Proficiency in free handling with different versions of applications of the generally-logical methods (GLM) for solving problems from different mathematical fields can be acquired in the process of continuous work for mastering certain fields of knowledge and skills. But this can be achieved in case of at least understood and remembered certain theoretical knowledge and skills the elements of which are included in the so called theoretical base of mathematical problems, as in the process of their further mastering they are applied together with the following units of knowledge and skills arranged according to their internal logical hierarchy:

(1) Knowledge and skills for conducting reasoning using synthesis on atomic and molecular level.

(2) Knowledge and skills for conducting reasoning using analysis by Pop's scheme and Euclid's scheme and on atomic and molecular level.

(3) Knowledge and skills for solving problems at the cellular level, where direct application of synthesis is appropriate.

(4) Knowledge and skills for solving problems where it is appropriate to apply consistently first analysis by Pop's scheme or by Euclid's scheme, and then – synthesis (although the analysis by the scheme of Pop has a probative value).

(5) Knowledge and skills for solving problems where it is appropriate the parallel application of analysis and synthesis, i.e. it is appropriate to apply sometimes synthesis, then analysis, until fully trace the way between the given and the argument (the sought) in the problem.

(6) Knowledge and skills for solving problems, where it is appropriate the joint application of analysis and synthesis, i.e. both are searching sufficient conditions for finding sought in the problem and necessary conditions – in order to effective use the hypotheses in the given (we will note that the strength of this version of the application of analysis and synthesis is the constant utilization the inner mathematical information of problem, determined from the links between the given and sought in it).

(7) Complex knowledge and skills (including the certain mathematical theory) to conduct analytical and synthetic activity for searching and implementation of solutions of different in content mathematical problems.

In accordance with the characteristics of the relation "reflection – synergetics" in the system "trained – trainer" these units can be organized into a system, the structure of which is represented by the model of fig. 1 in article[56] (Miloushev & Frenkev, 2007). Here we will just note that through knowledge and skills of units (1) and (2) are implemented activities on atomic and molecular level, while both the synthetic reasoning and the analytical ones (by Pop's scheme or by Euclid's scheme) may be of a type "chain" or a type "network". The activities on the "cellular" level, under which searching and realization of solutions to various mathematical problems in terms of self-organization is carried out, include knowledge and skills of units (3) – (6), namely: direct application of synthesis; consistent application first of analysis, then synthesis; parallel application of analysis and synthesis; joint application of analysis and synthesis. Conducting analytically-synthetic reasoning, applying complex knowledge and skills, including the relevant mathematical theory, not just knowledge about the nature and applicability of generally-logical methods for searching and realization of solutions of various in content mathematical problems, there are carried out activities on macro-level or on mega level.

[56]Milloushev, V. B. & Frenkev, D. G. (2007a). The system of activities for mastering generally-logical methods for solving mathematical problems in accordance with the principle of reflection. In: *„Didactics of Mathematics: Problems and Investigations"* (International Collection of Scientific Works), (28), (pp. 178-184). Donetsk, Ukraine: DonNU. [In Rus.]

We will note also that in the process of solving a variety of specific problems, in general, the knowledge and skills to implement the activities on atomic, molecular and cellular level (described from (1) to (6)) have the same structure. However, students usually have difficulties to notice this characteristic feature, as it was noted above, because the knowledge about generally-logical methods are closely related to other specific mathematical knowledge and moreover they are not provided in the curriculum for special study in high school. Therefore it is necessary, at certain moments, the teacher to indicate clearly the "community" of this structure, in order to accelerate the "detachment" of the common properties or characteristics of the knowledge and skills to implement the generally-logical methods from their specific carriers, i.e. in fact to take place the "*method of learning by summarizing reasoning*" on mastering the generally-logical methods.

An indicator of the maturity of knowledge and skills to implement the basic generally-logical methods is the belonging to the ZAD of the student of the last three units of knowledge and skills – (5), (6), (7), namely knowledge and skills, at least for parallel application of analysis and synthesis, and hence complex knowledge and skills (including the relevant mathematical theory) to conduct analytically-synthetic activity for search and realization of solutions to diverse in content mathematical problems.

We will note that, multiple transitions from analysis to synthesis and back, in the parallel, and even more – in their joint application for searching solutions of mathematical problems, essentially transfer the parallel and joint application of analysis and synthesis in productive *heuristic methods*[57] (Skafa & Milloushev, 2009), which are individual cases of the method "analysis through synthesis". They are basic means of differentiation of the problems-component, the solving of which plays a key role in the implementation of the solution of the original problem. Very often the problem-components are solved on the base of particularly-mathematical methods, mastering of which, in many cases, allows the one that solves the problem to use synthetic reasoning in the most part of the process of its solving. Similarly, we can say that the trend towards using some special heuristics in many cases is based on the skills about analytically-synthetic activity. Due to the outstanding advantages of these methods, their mastering must be the main objective of the education, and the development of skills for direct synthetic searching a solution and parallel implementation of analysis and synthesis help to accomplish the main goal. This determines the arrangement of problems in each subsystem as well as the sequence of subsystems in one whole system (thus the order of problems in different systems and subsystems turns in unison with the systemic principle about structural completeness).

In order to remove the random factor – "ignorance" of a certain theoretical unit (definition, theorem, formula, etc.), for example, due to forgetfulness, in the teaching practice, we recommend to draw up a list of the elements of the theoretical bases of the problems of a certain system (or subsystem, or test), then to arrange these elements thematically according to the curriculum, and before considering the respective problems of a given type to update the knowledge of learners about the theoretical elements from this list.

Taking into consideration all the foregoing, a sample system of problems should be developed and used in the training in order to update, supplement, reinforce and summarize the knowledge and skills of the students about basic generally-logical methods for solving problems. In the construction of such systems of problems there must be also taken into consideration the **objectives: a) – f)** described above. It is worth noting that the system of mathematical problems, developed in such a way, is playing the role of specific didactic tools for activation of reflection in teaching the solving

[57]Skafa, E. & Milloushev, V. (2009). *Constructing educational-cognitive heuristic activity in solving mathematical problems.* Plovdiv, Bulgaria: University press „Paisii Hilendarski". [In Bul.]

educational and criterial mathematical problems with a relatively high degree of complexity and problematicity.

The free handling with different variants of application of generally-logical methods for solving problems is the *criterion* for success of one education (see units of knowledge and skills (1) – (7), described above). This means that the knowledge of learners about the nature and applicability of these methods should be significantly improved, and must develop their skills for the effective and adequate use of the methods in solving problems of different mathematical areas. We have decomposed this criterion into the following specific types of skills:

1. Skills to perform analytically-synthetic activity:

a) skills to bring and formulate consequences through synthesis;

b) skills to bring and formulate consequences through the scheme of Euclid;

c) skills to find sufficient conditions through analysis by Pap's scheme;

d) skills to find necessary and sufficient conditions through parallel or joint application of analysis and synthesis.

2. Skills to differentiate problems-components.

3. Skills to solve mathematical problems truthfully, completely and justified.

While selecting and structuring the problems in the different systems the teacher should be guided by the following considerations: for each system of problems the students should be able to use all variants of generally-logical methods; the individual problems must not contain specific (not included in the school documentation) knowledge and particularly-methodical methods that could prevent the implementation of the basic generally-logical methods. Regarding the first consideration we will note that the different systems of problems, in their turn, must consist of subsystems for acquiring knowledge and skills, related to the application of the fundamental generally-logical methods in different versions. According to our experience, suitable subsystems are the following: **A** – subsystem of problems intended for introduction into the ZND of students relevant knowledge and skills for the generally-logical methods; **B** – subsystem of problems for introducing knowledge and skills about generally-logical methods in the ZAD of students; **C** – subsystem of problems for consolidation knowledge and skills to implement the generally-logical and private-mathematical methods in the ZAD; **D** – system of problems for maintenance complex knowledge and skills in ZAD and their transform into different situations.

We will note that in training the students consciously can be used one and the same problem to illustrate the application of different methods of solving problems (direct application of synthesis, sequential application first of imperfect analysis, then synthesis; ascending analysis; method of equivalence) in order more explicitly and clearly to highlight the characteristics and the differences between them and also to evaluate which method is more rational in the concrete case. The experience shows that it has a strong reflexive-training effect. In this regard, we will note that solving, for example, the problem of proving the inequality of Cauchy using the pointed different methods makes this inequality so popular for learners, that they accept it as a basic statement (a problem-theorem) and can use it ready-made to solve various problems. Thus a particular method can be differentiated – "method based on inequality of Cauchy" – for solving problems of certain types that can be arranged in specific systems or subsystems with effective multifunctionality (consolidation of knowledge and skills for applying one or more generally-logical methods, one or more particularly-mathematical methods, including the method that have been already separated, etc.). Moreover, the wide applicability of the inequality $\frac{a+b}{2} \geq \sqrt{ab}$ gives the learners motivation to get successively acquainted with its summary for three or more non-negative integers: $\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}$ (where $n \in \mathbb{N}$, a $a_1,\ a_2, \ldots, a_n$ are arbitrary non-negative integers), and also with some of its applications. The realization of all this is according to the scheme-model for the acquisition and application of knowledge based on the relationships between intellectual and praxeological reflection (Fig. 1 in the paper[58] (Milloushev & Frenkev, 2008)).

In order to create opportunities the students to transfer their skills to perform both imperfect analysis and ascending analysis in solving geometry problems as well (not only for proof and calculation, but also for construction) and to convince them that for such problems it is even more productive, it is appropriate in the subsystem of problems **A**, designed to master these methods, to include the following problems, because they allow solutions in different ways in each of which the Pop's scheme is applicable. At the same time, we note that these problems from the subsystem **A**, except for mastering the generally-logical methods are intended for insertion into the ZND of students their ability to perform the more complex activity – *parallel application of analysis and synthesis*.

Problem A1. It is given an isosceles triangle ABC $(AC = BC)$. If $tg\,\gamma = \frac{3}{4}$, where $\gamma = \angle ACB$ (Fig. 3) prove that the medians $A_1(A_1 \in BC)$ and $BB_1(B_1 \in AC)$ are orthogonal.

Since this problem is first, in the solving of which (in the practical work with students) is used the parallel application of analysis and synthesis, the methodical activity for the searching of its solution with students is appropriate to conduct in the following ***plan***:

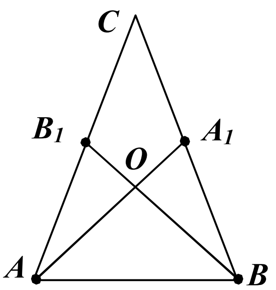


Figure 3.

1. <u>*Update*</u> knowledge about key elements of the theoretical basis of the problem – the formula for median in the triangle, cosine theorem, characteristic features of the term of a right triangle. It is also proved useful to remind knowledge about the method of parametrization.

2. *Guidance* to appropriate version of application of analytically-synthetic reasoning in the searching of solution. For this purpose it constructed the scheme on Tab. 1. So the objects of mental activity are separated. From the scheme it is noticeable that it's difficult using only synthesis to get from the given in the problem to the sought, and also only through analysis to get from the sought to the given, i.e. as in this case is not appropriate to use only one of these versions of the application of analysis and/or synthesis. It is therefore appropriate to use another option for searching a solution, for example, the parallel application of analysis and synthesis (Tab. 1).

3. *Detailing* the stages and steps of the process of searching a solution in terms of the outlined, in the previous stage, version of application of the basic generally-logical methods. We will note that what is characteristic for this problem is that in the course of applying these methods additional information is produced, on the base of which "on the go" new ideas to searching a solution occur.

The proof that the medians AA_1 and BB_1 are perpendicular to each other can be realized in two different ways. In the first way it is enough to be parameterized to identity ΔAOB, white in the other one – ΔAOB_1 (or ΔBOA_1). Here we will present the finding out of the solution of the problem schematically only in the first way in parallel application of analysis and synthesis.

Solution (in the first way). For brevity, we present schematically only the result of the analytically-synthetic activity of the students that are carried out (with the help of the tutor).

[58]Milloushev, V. B. & Frenkev, D. G. (2008). About one reflexive model of education and its application. In: *Mathematics and Mathematical Education,* (pp. 61-72). Sofia, Bulgaria: BAS. [In Bul.]

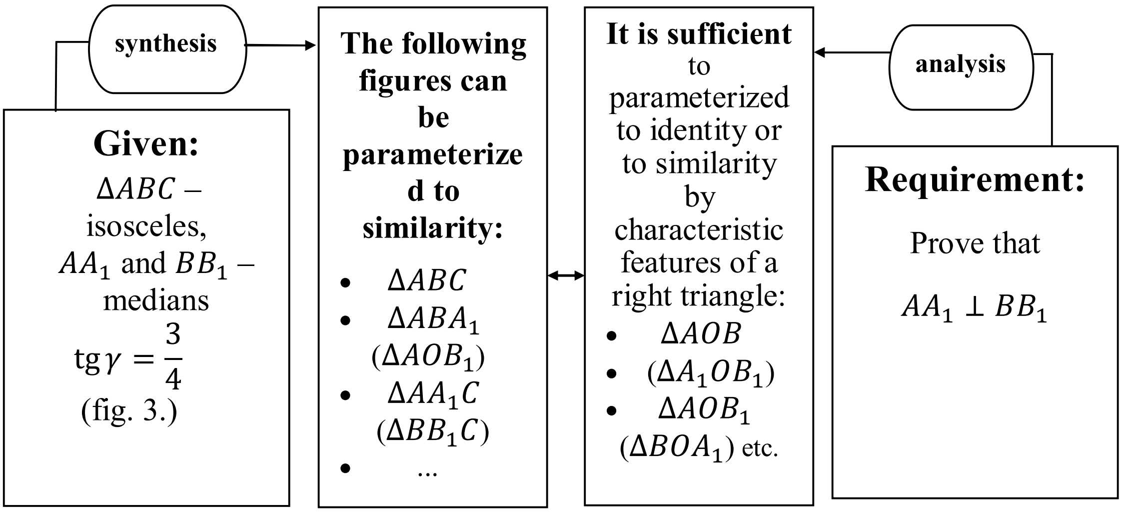


Table 1. Reasoning in parallel application of synthesis and analysis

Ascending analysis – I step: $AA_1 \perp BB_1 \Leftarrow \angle AOB = 90^\circ \Leftarrow AB^2 = AO^2 + BO^2$, (where O is the intersection point of the medians).

Synthesis – I step. From AA_1 and BB_1 – medians $\Longrightarrow AO = \frac{2}{3}AA_1$, $BO = \frac{2}{3}BB_1$,

$AA_1 = \frac{1}{2}\sqrt{2c^2 + a^2}$ and $BB_1 = \frac{1}{2}\sqrt{2c^2 + a^2}$ (where *a* is the leg, and *c* is the base of ΔABC) $\Longrightarrow AO = \frac{1}{3}\sqrt{2c^2 + a^2}$ and $BO = \frac{1}{3}\sqrt{2c^2 + a^2}$.

Ascending analysis – II step: $AB^2 = AO^2 + BO^2 \Leftarrow c^2 = \frac{1}{9}(2c^2 + a^2) + + \frac{1}{9}(2c^2 + a^2) \Leftarrow 9c^2 = 4c^2 + 2a^2 \Leftarrow 5c^2 = 2a^2$.

Synthesis – II step. From $\mathrm{tg}\gamma = \frac{3}{4} \Longrightarrow \cos\gamma = \frac{4}{5}$, and according to the cosine theorem:

$c^2 = 2a^2 - 2a \cdot a \cdot \frac{4}{5} \Longrightarrow 5c^2 = 2a^2$.

In order to create conditions for developing skills for intellectual and praxeological reflection the students are offered to compare the reasoning incurred so far with the reasoning in previous problems, where there have been applied only synthesis, or only analysis, or sequential first analysis, then synthesis , and to show the similarities and differences between them. Thus they note, that for the finding of the solution now, hardly can be applied only analysis or only synthesis. Otherwise, it is applied now analysis, then synthesis, therefore, to allow formation of a synthetic solution should be possible, following the appropriate symbols for short recording ($\Leftarrow$ and $\Longrightarrow$), eventually, "meeting" of the analytical and synthetic reasoning. Since this "meeting" exists, it can be done the following

4. *Synthetic form of solution.*

From $tg\gamma = \frac{3}{4}$, $tg\gamma = \frac{sin\gamma}{cos\gamma}$ and $sin^2\gamma + cos^2\gamma = 1 \Longrightarrow cos\gamma = \frac{4}{5} \Longrightarrow c^2 = a^2 + + a^2 - 2a \cdot a \cdot \frac{4}{5} \Longrightarrow 5c^2 = 2a^2 \Longrightarrow 9c^2 = 4c^2 + 2a^2 \Longrightarrow c^2 = \frac{1}{9}(2c^2 + a^2) + + \frac{1}{9}(2c^2 + a^2)$.

On the other hand, since AA_1 and BB_1 are medians, the equations follows:

$AO = \frac{2}{3}AA_1$, $BO = \frac{2}{3}BB_1$, $AA_1 = \frac{1}{2}\sqrt{2c^2 + a^2}$ and $BB_1 = \frac{1}{2}\sqrt{2c^2 + a^2} \Longrightarrow$

$AO = \frac{1}{3}\sqrt{2c^2 + a^2}$ and $BO = \frac{1}{3}\sqrt{2c^2 + a^2} \Longrightarrow AO^2 = \frac{1}{9}(2c^2 + a^2)$ and $BO^2 = = \frac{1}{9}(2c^2 + a^2)$.

Using the last two equalities and the equality:

$c^2 = \frac{1}{9}(2c^2 + a^2) + \frac{1}{9}(2c^2 + a^2)$, follows, that $AB^2 = AO^2 + BO^2$.
Therefore ΔABO is a rectangular with straight angel in the apex O, i.e. $AA_1 \perp BB_1$.

This problem is solved.

In the context of the idea about "aggregate theoretical units" (by Erdnievi), also with a view to developing skills for self-awareness (intellectual reflexive skills) and self-organization in the stage "A look backwards" the students are suggested to examine also the question of whether the opposite statement is true. With their active participation in formulating and solving in a similar way an opposite problem of the given one, which requires to put the requirement to formulate also a problem that joins together these two problems. Here we will present one of the inverse problems created by the students and its solution based on its general information.

Problem A**2.** It is given an isosceles triangle ΔABC (*AC*=*BC*). If the medians $AA_1(A_1 \in BC)$ and $BB_1(B_1 \in AC)$ are orthogonal each other, find out that $\cos\gamma$, where $\gamma = \angle ACB$.

Solution. From $AA_1 \perp BB_1 \Longrightarrow \angle AOB = 90° \Longrightarrow AB^2 = AO^2 + BO^2$, (where O is the intersection point of the medians. Because AA_1 and BB_1 are medians, then point O is a centroid. Then the following equations are fulfilment $AO = \frac{2}{3}AA_1$, $BO = \frac{2}{3}BB_1$, $AA_1 = \frac{1}{2}\sqrt{2c^2 + a^2}$ and $BB_1 = \frac{1}{2}\sqrt{2c^2 + a^2}$ (where *a* is the leg, and *c* is the base of ΔABC). Therefore

$AO = \frac{1}{3}\sqrt{2c^2 + a^2}$ and $BO = \frac{1}{3}\sqrt{2c^2 + a^2} \Longrightarrow AO^2 = \frac{1}{9}(2c^2 + a^2)$ and $BO^2 = = \frac{1}{9}(2c^2 + a^2)$.

Then from the equality $AB^2 = AO^2 + BO^2$ is obtained that $5c^2 = 2a^2$. Using this result and the cosine theorem about ΔABC, is found out that $cos\gamma = \frac{4}{5}$.

It is appropriate to develop a second way to solve the problem using the specific information in it, namely the fact that ΔABC is an isosceles one. Then if we build the third median CC_1, it follows that it is both a height and bisector in the triangle. Therefore $tg\frac{\gamma}{2} = \frac{AC1}{CC1}$. But $AC_1 = OC_1$, that is why $tg\frac{\gamma}{2} = \frac{OC1}{CC1} = \frac{1}{3}$. From this and with the formulas $cos\gamma = \frac{1-tg^2\frac{\gamma}{2}}{1+tg^2\frac{\gamma}{2}}$ is received, that $cos\gamma = \frac{4}{5}$.

Usually most students qualify this solution (this way) as shorter than the first one.

In the stage "A look backwards" it is useful a third way for solving the problem to be considered, based on vector method. Suitable is a vector base composed of vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$, because their scalar product contains $cos\gamma$. Without limitation of the community, it can be assumed that the basis vectors are single. They are used to express the vectors $\overrightarrow{AA_1}$ and $\overrightarrow{BB_1}$, then is used that scalar product of them is equal to zero, since they are perpendicular. From the resulting equality is found out that $cos\gamma = \frac{4}{5}$. It is worth noting also that the vector method can be applied to solve the problem 1.

The question which of these three ways to solve the problem is more rational is discussed with the students, which helps to develop their reflexive skills.

In order to create opportunities those students to show their skills to perform an analysis by the scheme of Euclid in solving geometric problems, it's appropriate to consider the problems appearing representatives of the subsystem **B**, the purpose of which is inserting the relevant knowledge and skills of students in their ZAD. Having in mind this and also the fact that synthetic reasoning are conducted in combination with analytical, here we will focus on the methodology of work on solving some problems of the subsystem **B**. The work on other problems of this subsystem is conducted similarly, and therefore we will not describe it here.

Problem B1. If *a, b* and *c* are sides of the triangle *ABC* and they satisfy the equality $a^2 + b^2 = 5c^2$, prove that its medians AA_1 and BB_1 are orthogonal.

Usually in the initial stage of searching a solution to a given problem, it is difficult for the students to choose the appropriate generally-logical method for its "attack". In the initial processing of the information in this problem there is noticed that it would be very difficult to found a solution through direct application of synthesis, since it is not clear how to use the given equality $a^2+b^2=5c^2$. On the other hand, there are many indications of perpendicularity of two lines, as well as many consequences of the statement that two lines are perpendicular, so when choosing an analytical method is preferable to "go forward" through the scheme of Euclid, "throwing" continually look at the given in the problem. We will note, of course, that there could be applied reasoning by the Pop's scheme, but here, in order to improve the skills to use the scheme of Euclid, we direct students to apply imperfect analysis. With their active participation and relevant proposals of some of them, is realized the following

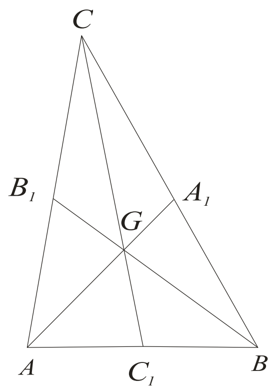


Figure 4.

I*mperfect analysis.* Let suppose that medians AA_1 and BB_1 are orthogonal each other and G is their point of intersection, i.e. G is the centroid of the triangle ΔABC (Fig. 4). Then ΔABG is a rectangular one. Therefore, if CC_1 is the third median of ΔABC, then GC_1 is the median to the hypotenuse in rectangular ΔABG and then $GC_1 = \frac{1}{2}AB = \frac{c}{2}$. But from the feature of centroid, follows that $CC_1 = 3.GC_1 = \frac{3c}{2}$. Thus the sides of ΔBCC_1 are expressed by the parameters *a* and *c* and therefore it is appropriate to apply the cosine theorem, i.e.:

$$cos\beta = \frac{BC^2 + BC_1^2 - CC_1^2}{2 \cdot BC \cdot BC_1} = \frac{a^2 + \frac{c^2}{4} - \frac{9c^2}{4}}{2 \cdot a \cdot \frac{c}{2}} = \frac{4a^2 - 8c^2}{4ac} = \frac{a^2 - 2c^2}{ac},$$

$$cos\beta = \frac{a^2 - 2c^2}{ac}. \qquad (1)$$

Similarly, by the cosine theorem about ΔABC, follows that:

$$cos\beta = \frac{a^2 + c^2 - b^2}{2ac}. \qquad (2)$$

Then from the equations (1) and (2) is satisfied that:

$$\frac{a^2 - 2c^2}{ac} = \frac{a^2 + c^2 - b^2}{2ac} \Rightarrow$$
$$\Rightarrow 2a^2 - 4c^2 = a^2 + c^2 - b^2 \Rightarrow$$
$$\Rightarrow a^2 + b^2 = 5c^2.$$

But it is the given inequality. Therefore the analysis is completed.

Solution (*synthesis*). Starting from the given equality $a^2+b^2=5c^2$, being the last step of the analysis that was carried out, **doing in reverse order the reverse operations**, consistently is received:

$$a^2 + b^2 = 5c^2 \ /+a^2 - b^2 - 4c^2$$
$$\Rightarrow 2a^2 - 4c^2 = a^2 + c^2 - b^2 - 4c^2 \ /:2ac > 0$$
$$\Rightarrow \frac{a^2 - 2c^2}{ac} = \frac{a^2 + c^2 - \mathrm{b}^2}{2ac}.$$

But about ΔABC is satisfied that:

$$\frac{a^2 + c^2 - b^2}{2ac} = cos\beta.$$

Then from the last two equalities follows that:

$$\frac{a^2-2c^2}{ac}=cos\beta. \qquad (3)$$

On the other hand, according to the cosine theorem about ΔBCC_1 is fulfilled

$$cos\beta=\frac{a^2+\frac{c^2}{4}-CC_1^2}{2.a.\frac{c}{2}}=\frac{4a^2+c^2-4.CC_1^2}{4ac}. \qquad (4)$$

From (3) and (4) follows the equality:

$$\frac{a^2-2c^2}{ac}=\frac{4a^2+c^2-4.CC_1^2}{4ac}.$$

By getting rid of the denominators is obtained successively:

$$4a^2-8c^2=4a^2+c^2-4.CC_1^2 \implies 4.CC_1^2=9c^2 \implies \mathrm{CC_1}=\frac{3\mathrm{c}}{2},$$

because $CC_1>0$.

Because G is a centroid of ΔABC, it is executed that:

$$GC_1=\frac{1}{3}CC_1=\frac{1}{3}.\frac{3c}{2}=\frac{c}{2} \implies GC_1=\frac{1}{2}AB.$$

But GC_1 is a median in ΔABG, and the last equality is characteristic for the triangle to be rectangular. Therefore $\angle AGB = 90°$, i.e. the medians AA_1 and BB_1 are perpendicular to each other.

In the stage "A Look Backwards" the question of finding other solutions using other theoretical basis is discussed first. Some students are directed to use the formula for median:

$$AA_1=\tfrac{1}{2}\sqrt{2b^2+2c^2-a^2}\ ,\ BB_1=\tfrac{1}{2}\sqrt{2a^2+2c^2-b^2}\ .$$

Developing their idea, these students realize the analysis as follows.

Let the medians AA_1 and BB_1 are orthogonal and G is their point of intersection, i.e. G is the centroid of triangle ΔABC (Fig. 4). Therefore ΔABG is rectangular. Then the equality of Pythagoras is fulfilled $AG^2+BG^2=AB^2$. But:

$$AG=\tfrac{2}{3}\cdot AA_1=\tfrac{2}{3}\cdot\tfrac{1}{2}\sqrt{2b^2+2c^2-a^2}=\tfrac{1}{3}\sqrt{2b^2+2c^2-a^2},$$

and

$$BG=\tfrac{2}{3}\cdot BB_1=\tfrac{2}{3}\cdot\tfrac{1}{2}\sqrt{2a^2+2c^2-b^2}=\tfrac{1}{3}\sqrt{2a^2+2c^2-b^2}.$$

By substituting AG and BG in the equality of Pythagoras, consequently is received:

$$\frac{1}{9}(2b^2+2c^2-a^2)+\frac{1}{9}(2a^2+2c^2-b^2)=c^2 \implies$$
$$\implies 2b^2+2c^2-a^2+2a^2+2c^2-b^2=9c^2 \implies$$
$$\implies a^2+b^2+4c^2=9c^2 \implies$$
$$\implies a^2+b^2=5c^2.$$

The last equality is given in the condition of the problem.

In accordance with the analysis that was made, the new solution is formed:

Solution (*synthesis*). We starts again from the given inequality $a^2+b^2=5c^2$ and to both sides is added the expression $4c^2$, which in the left side is written as a sum: $2c^2+2c^2$. The equality is obtained $b^2+2c^2+a^2+2c^2=9c^2$. Then in the left side of this equality are added and subtracted a^2 and b^2 in order to obtain the expressions involved in the formulas for the medians AA_1 and BB_1, namely

$$(2b^2+2c^2-a^2)+(2a^2+2c^2-b^2)=9c^2. \qquad (5)$$

Since the formulas about the medians can be written in the following way:

$(2\cdot AA_1)^2=2b^2+2c^2-a^2$ and $(2\cdot BB_1)^2=2a^2+2c^2-b^2$,

then the equality (5) takes the form $(2\cdot AA_1)^2+(2\cdot BB_1)^2=9c^2$.

The last equality can be presented in the form:

$$\left(\frac{2}{3}AA_1\right)^2+\left(\frac{2}{3}BB_1\right)^2=c^2.$$

But $\frac{2}{3}AA_1 = AG$ and $\frac{2}{3}BB_1 = BG$.

Therefore $AG^2 + BG^2 = AB^2$. From here, according to the converse theorem of Pythagoras, follows that ΔABG is rectangular, as $\angle AGB = 90^\circ$. Therefore $AG \perp BG$, i.e. the medians AA_1 and BB_1 are orthogonal. The problem is solved in a second way as well.

On the next place in the stage "A look backwards" we put the question whether the opposite statement is true. With the active participation of students the inverse problem of the given one is formulated and solved in a similar way, this needs to put the requirement to formulate also a problem that brings together these two problems. Another didactic task should be placed, to solve the last problem by ascending analysis, which helps students to self-aware once again the similarities and differences between the two types of analysis. Thus a rich complex activity is accomplished which includs solving the problem in several ways using different theoretical basis, construction of inverse problems and application again the method that is used, and finally, uniting in a synergistic aspect the two problems in one, the solution of which by Pop's scheme is natural for the students. The realization of such a complex activity with applying many times the scheme of Euclid in one and the same geometric situation, contributes significantly to the further self-improvement the skills of students for reasoning on this scheme and inserting them in their ZAD. All this is a prerequisite for a more active and independent participation of students in solving the next problems of the subsystem B. Here, however, we will present what it is stated in another problem only (given on a competitive entrance examination for the university) of this subsystem, and also a short comment on the determination of its structure and ideas to solve it. The solutions themselves, with possibilities of transformation and creation new problems the reader can find in our article[59] (Milloushev & Frenkev, 2008).

Problem B2. Find the acute angles of a rectangular triangle if the radius of the inscribed in the triangle circle and radius of the described around it circle refer as: a) 2 : 5; b) 1 : $(\sqrt{3} + 1)$.

In the process of *realizing the given in the problem and determining its structure* using the traditional indications of a rectangular triangle: a and b – cathetus, c – hypotenuse, α and β – the acute angles of the triangle, r – radius of the inscribed circle, R – radius described circle, the students come to the following conclusions.

Since the rectangular triangle is determined to identity with two parameters, of which at least one is metric (length of a segment or an area), and only the attitude of r and R are given in the problem, then the figure is *determined to similarity* – this is the structure of the problem. This information leads us to the following famous general ideas for solving:

1) introducing additional metric parameter and reducing to a problem in which the figure is determined to identity;

2) using the method of similarity;

3) as in the problem an invariant of similarity – angle is sought, it is probably possible to create a trigonometric equation about it.

These ideas, in turn, determine appropriate strategies for searching solutions. These strategies are realized in the paper cited above, where opportunities for transformation and creation new problems are shown.

In stage "A look backwards" there is made an evaluation about the rationality of the three ways for solving the problem. In this connection, the students noted that last idea leads to the most rational solution while solving it using ideas 1) or 2) needs much time,

[59] Milloushev, V. B. & Frenkev, D. G. (2007b). A system vertions of transformation mathematical problems - means of intensifying of reflection. *In: "Science, Education and Time as a Care".* Reports of the Anniversary Scientific Conference with International Participation, 30.11.-1.12.2007, Smolyan, Bulgaria. (pp. 127-133). [In Bul.]

because it leads to solving systems of equations, and even more in b) the system is with irrational coefficients.

In order to develop praxeological reflection in students, it is appropriate to assign them to do self-analysis and self-assessment of the activity of using their knowledge and skills to deal with the problem, set in the didactic task for transformation and creation new problems. In this connection, the students said that for this purpose they need skills to solve geometrical problems by combined application of ascending analysis and through creating and solving trigonometrical equation. Then we announce that often in solving mathematical problems, trigonometric equations have to be created and solved, and therefore can be "differentiated" the so called "trigonometric method." It is a particularly-mathematical method because it is based on specific mathematical knowledge, and this in turn means that it can be seen as part of the theoretical basis of the problem, that is why it (as well as any other particialar method also based on certain specific mathematical knowledge) it is applied together with generally-logical methods. Such similar complex activity students perform on mega-level later in dealing with the problems of system **D**.

The system **C** includes problems that are designed to *improve* and *maintain* in ZAD of students their knowledge and skills, relating to the implementation of generally-logical methods in various versions. The maintenance of knowledge and skills in ZAD of students is realized primarily through individual work with the problems of the system **C**. But after its implementation, with some problems that have more complex structure of the solutions, require the students to present their "training solutions", which, in terms of the effectiveness of intellectual reflection, it appears to be very useful. We shall not describe in details the methodology used to work, but will mention only individual problems-representatives of its subsystems, accompanied by "training solutions developed by the students and edited by us.

Problem C1. If a, b and c are the lengths of the sides of any triangle, prove the inequality $a^2 + b^2 + c^2 < 2(ab + ac + bc)$.

Solution. Because the sides of the triangle are equal, without breaking the community, it can be assumed that $a \geq b \geq c$. Then, according to the theorem of inequalities between the sides of the triangle, is fulfilled that $a - b < c$, $b - c < a$, $a - c < b$.

Since the right sides of these three inequalities are positive (Why?), and the left ones are non-negative, they can be raised in a square in which the direction of the inequalities is the same, i.e. follow the inequalities:

$a^2 + b^2 - 2ab < c^2,\ b^2 + c^2 - 2bc < a^2,\ a^2 + c^2 - 2ac < b^2.$

The inequalities are unidirectional. Therefore they can be summed member by member, as a result of which we have:

$2a^2 + 2a^2 + 2a^2 - 2ab - 2bc - 2ac < a^2 + b^2 + c^2 \ \Rightarrow$

$\Rightarrow\ 2a^2 + 2a^2 + 2a^2 - a^2 - b^2 - c^2 < 2ab + 2bc + 2ac \ \Rightarrow$

$\Rightarrow\ a^2 + b^2 + c^2 < 2(ab + ac + bc).$

The inequality is proved.

Problem C2. If a and b are the lengths of cathetus, c is the length of hypotenuse of a rectangular triangle so that, $b + c \neq 1$, $c - b \neq 1$, then prove the equality:

$log_{b+c}a + log_{c-b}a = 2 \cdot log_{b+c}a \cdot log_{c-b}a.$

Assimilation of the given in the problem and orientation to a strategy for searching a solution. It is clear from the given in the problem, that $a > 0$, $b > 0$, $c > 0$, $c - b > 0$. When $a = 1$ the equality is obviously satisfied. Let now $a \neq 1$. From the fact that the triangle is a rectangular one the equality of Pythagoras $a^2 + b^2 = c^2$ can be written, but still it is not clear exactly what transformations to perform with it. That's why it is therefore appropriate to apply first.

Imperfect analysis. If we change when $a \neq 1$, the bases of the logarithms in the equality $log_{b+c}a + log_{c-b}a = 2 \cdot log_{b+c}a \cdot log_{c-b}a$, that have to be proved, is consequently obtained:

$$\frac{1}{log_a(b+c)} + \frac{1}{log_a(c-b)} = \frac{2}{log_a(b+c).log_a(c-b)} \Rightarrow$$

$\Rightarrow\ log_a(c-b) + log_a(b+c) = 2 \ \Rightarrow$

$\Rightarrow\ log_a(c-b)(c+b) = \ log_a a^2 \ \Rightarrow$

$(c-b)(c+b) = a^2 \ \Rightarrow\ c^2 - b^2 = a^2 \ \Rightarrow\ c^2 = a^2 + b^2.$

Therefore the triangle with sides *a*, *b* and *c* is a rectangular one with a hypotenuse *c*.

After conducting this imperfect analysis, now it is clear how to perform the synthetic solution.

Solution. Since *a* and *b* are cathetus, and *c* is a hypotenuse in a rectangular triangle, then the following equality is fulfilled

$c^2 = a^2 + b^2 \ \Rightarrow\ c^2 - b^2 = a^2 \ \Rightarrow\ (c-b)(c+b) = a^2.$

Taking logarithms at both sides of the equality on base *a*, under the constraints mentioned above, it is received that

$log_a(c-b)(c+b) = log_a a^2 \ \Rightarrow\ log_a(c-b) + log_a(b+c) = 2.$

By dividing both sides of the last equality with the expression

$log_a(c-b) \cdot log_a(b+c) \neq 0$, we have

$$\frac{1}{log_a(b+c)} + \frac{1}{log_a(c-b)} = \frac{2}{log_a(b+c).log_a(c-b)} \Rightarrow$$

$log_{c+b}a + log_{c-b}a = 2 \cdot log_{b+c}a \cdot log_{c-b}a.$

Problem C3. The base of pyramid is a rectangular triangle with an area *Q* and acute angle *α*. The lateral face that goes along the cathetus subtended by the given angle is perpendicular to the plane of the base, and the other two lateral faces conclude with the plane of the base angles equal to *β*.

a) Find the volume of the pyramid;

b) Find the tangent of the angle between the lateral edge, passing through the vertex of the given angle, and the plane of the base. (The problem is given on a competitive entrance examination, Veliko Turnovo University, 1991).

The fact that the given two lateral faces conclude with the plane of the base, and not with the base angles, angels equal to *β*, requires consideration of two geometric situations. Here we present a "teaching solution" of the problem in the case, when the apex of the pyramid is above the interior of the base. In the other case, when the apex of the pyramid is above the exterior of the base, in the plane of the base, the reasoning is similar.

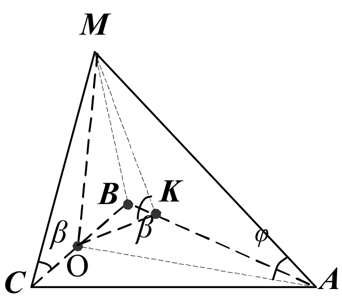


Figure 5.

Assimilation of the given in the problem and motivation of the figure. Let's the pyramid is $MABC$, and its base is the rectangular triangle ABC with a right angle in the vertex C and $\angle BAC$=α (Fig. 5). Since the lateral face BCM is perpendicular to the base of the pyramid, then the orthogonal projection O of the apex *M* on the plane of the base lies on their intersection line *BC*. On the other hand, since the two lateral faces ACM and ABM conclude one and the same angle *β* with the base, then O lies also on the bisector of $\angle BAC.$ Because $AC \perp CO$ and CO is the orthogonal projection of *CM* on the plane *ABC*, then according to the theorem about the three perpendiculars follows that $AC \perp CM$, is fulfilled i.e. $\angle MCO = \beta$ is the linear angle of the dihedral angle between the planes ACM and *ABC*. The angle the tangent of which is searched in problem b), is $\angle MAO.$ Let's denote it with φ.

We will note that the motivation and the construction of the figure (as important, although auxiliary, models of the "*material model*"[60] (Frenkev, Milloushev & Boykina, 2007) of the process of solving the problem) is carried out by partial synthetic reasoning using specific knowledge (knowledge about the properties of perpendicular planes, about the property of lateral faces concluding equal angles to the plane of the base, the theorem of the three perpendiculars etc.).

Ascending analysis. a) In order to find the volume of the $MABC$, having in mind that the area Q of its base is given, it is enough to find the length of the height MO. In order to find MO, having in mind the fact that the triangle MOC is a rectangular one and as $\angle MCO = \beta$ is known, it is sufficient to find the length of CO. In order to find CO, considering that it is cathetus in the rectangular ΔACO, about which $\angle OAC = \frac{\alpha}{2}$ is known, it is enough to find the length of the cathetus AC. But AC is cathetus in the rectangular triangle ΔABC, which is parametrized to identity with the given elements $\angle BAC = \alpha$ and Q. This leads to a "complex" but of known type a problem-component, the finding a solution of which is implemented through the following reasoning. In order to find AC, considering that the face Q of the triangle ABC is given, it is sufficient to find the length of its other cathetus BC or to compose an equation about AC. But, as $\angle BAC = \alpha$ is known, then using the link $BC = AC \cdot tg\alpha$, leads to the equality $Q = \frac{1}{2} AC \cdot BC = \frac{1}{2} AC^2 \cdot tg\alpha$, from which AC can be expressed by the given elements $\angle BAC = \alpha$ and the face Q of the base of the pyramid.

The analysis by the Pop' scheme of the problem a) is completed. We will not present the synthetic form of the solution.

b) In order to find $tg\varphi$, having in mind, that we have already known the length of the height MO, it is sufficient to find the length of AO. Thus we also reach a problem-components of a known type - to find AO is enough to consider triangle ACO, in which the two cathetus and $\angle OAC = \frac{\alpha}{2}$ are known.

We are not going to present the synthetic solution again.

In solving the next problem of subsystem **C** some students conduct first synthetic reasoning (it is clear that such skills dominate in them), and other students – analytical reasoning.

Problem C4. Prove that, if $log_k x$, $log_m x$, $log_n x$ are consecutive members of arithmetic progression and *k, m, n, x* are positive numbers, different from 1, and then the equality:

$n^2 = (kn)^{log_k m}$ is satisfied.

We will note that this problem is relatively easy, because its statement creates prerequisites for parallel application of basic generally-logical methods by applying *synthesis* first and its theoretical basis includes familiar algebraic material. From this perspective, it serves for "self-updating" the students' skills for parallel application of synthesis and analysis in the course of individual solving the problems of the subsystem. This makes them easier for the search of solutions (based on the same approach) in the next problems of the subsystem.

Synthesis. Because there is given that $log_k x$, $log_m x$, $log_n x$ are consecutive members of arithmetic progression, then according to its characteristic property the following equality is satisfied $2 \cdot log_m x = log_k x + log_n x$. Since the three logarithms have different bases, it is appropriate to go to one and the same base. What can it be? The answer of this question is included in the thesis – the equality, which must be proved. (Essentially, analytical reasonings appear here, though implicitly). The thesis reminds

[60]Frenkev, D. G., Milloushev, V. B., & Boykina, D. V. (2007). A complex model of the process of solving mathematical problems of a certain type. In: *Mathematics and Mathematical Education,* (pp. 429-435). Sofia, Bulgaria: BAS. [In Bul.]

that it must be gone to the base k. Then the last equality accepts the following type $2 \cdot \frac{log_k x}{log_k m} = log_k x + \frac{log_k x}{log_k n}$.

In this equation the common multiplier $log_k x$ is different from 0, because it is given that $x \neq 1$. Therefore, it can be reduced with this common multiplier and, after releasing from the denominator the equality $2 \cdot log_k n = log_k m \cdot log_k n + log_k m$ is obtained.

If the student guesses that the right side of the equality could be decomposed in multipliers, he will write and equality $2 \cdot log_k n = log_k m(log_k n + 1)$, if he does not think of this transformation, his synthetic reasoning stops on the previous equality. Our long experience as university tutors (and teachers) shows that usually high school students, and also some university students stop there, because they do not think to replace the number 1 in brackets with logarithm. Then of course you have to start analytical reasoning, starting from the thesis, ie to jump from synthesis to analysis.

Imperfect analysis. Taking logarithms of both sides of the equality $n^2 = (kn)^{log_k m}$ to the base k, is received

$log_k n^2 = log_k (kn)^{log_k m} \Rightarrow$
$\Rightarrow 2.log_k n = log_k m.log_k (kn) \Rightarrow$
$\Rightarrow 2.log_k n = log_k m(log_k k + log_k n) \Rightarrow$
$2 \cdot log_k n = log_k m(1 + log_k n)$.

The final step in this analytical reasoning coincides with the last step of the synthetic one. That's why the "meeting" of two chains of reasoning is done. As the analysis that is carried out, however is imperfect, it is obligatory to perform a replacement of this part of the recording with synthetic reasoning.

Continuing the synthesis:

From the equality $2 \cdot log_k n = log_k m(log_k n + 1)$, by replacing the number 1 with $log_k k$, the following statement $2.log_k n = log_k m(log_k n + log_k k)$ is received.

From it, using antilogarithm follows the equalities:

$log_k n^2 = log_k m.log_k (nk) \Rightarrow$
$\Rightarrow log_k n^2 = log_k (kn)^{log_k m} \Rightarrow$
$\Rightarrow n^2 = (kn)^{log_k m}$.

The problem is completed.

Problem C5. In the acute-angled triangle ABC (Fig. 6) there are known the lengths of: side $AC = 50$, the median $AM = 40$ and the orthogonal projection AH of AC upon AB, $AH = 14$. Prove that $\angle AMH = \angle ACH$.

Ascending analysis: In order to prove that $\angle AMH = \angle ACH$, having in mind the fact that their sides pass through the ends of the segment AH, it is sufficient to describe a circle around the rectangle $AHMC$. In order to prove that a circle can be described around $AHMC$, knowing that angle AHC is right and using the same criterion as for the inscribed quadrangle in a circle, it is enough the equality $\angle AMC = \angle AHC$ to be fulfilled. To prove the equality of these angles as we mean that one is right ($\angle AHC = 90°$), it is sufficient to prove that $\angle AMC = 90°$. To prove that $\angle AMC$ is right, it is sufficient to establish that AM is a height to the side BC in the triangle ABC. In order to prove that AM is a height, knowing that it's a median (by the given) in the triangle ABC, it is enough to prove that this triangle is an isosceles one, as $AB = AC$. In order to prove that $AB = AC$, it is sufficient to establish that $AB = 50$ (because $AC = 50$).

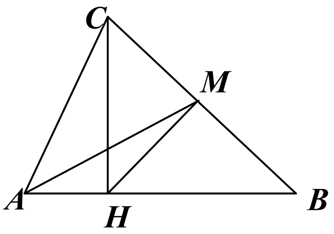


Figure 6.

Synthesis:

Let's denote AB with x. Since $AH = 14$ and ΔABC is acute-angled, then $HB = x - -14$. Then from the rectangular ΔAHC, by the theorem of Pythagoras, we find out that

$HC = 48$, and from ΔBHC we have $BC^2 = 2304 + (x - 14)^2$. As replacing the given elements and those that have been already expressed by x in the formula for median $AM = \frac{1}{2}\sqrt{2 \cdot AB^2 + 2 \cdot AC^2 - BC^2}$, we receive the equation:

$$40= \frac{1}{2}\sqrt{2x^2 + 5000 - 2304 - (x - 14)^2}.$$

From it we find that $AB = 50$. Therefore the "meeting" of analysis and synthesis is performed. To the foregoing part of the synthetic solution we will add the following. From $AB = AC = 50$ follows that ΔABC is isosceles. So there its median AM is also a height, i.e. $\angle AMC = 90°$. But it is given that $\angle AHC = 90°$. Then the segment AC is seen at a right angle from the points M and H, which means that around the rectangle $AHMC$ can be described a circle (with diameter AC). Therefore $\angle AMH = \angle ACH$.

The following problems are from subsystem **D,** designed for periodic updating and consolidation of knowledge and skills (related to implementation in reflexive-synergetic plan, the generally-logical methods for solving problems together with the mastering other methods and means), their maintenance in the ZAD of students and transformation in different situations.

We will emphasize that the majority of students are aware of the role of generally-logical methods as a means of self-organization in the course of solving the problems of the system **D**. Here we present just some of these problems (which we have provided to prospective students for individual work) and partial comments on their characteristics or the methods of their solving without a full description of the solutions of the problems.

Problem D1. If the numbers $a > 0$ and $b > 0$ satisfy the equality $13ab = 4a^2 + 9b^2$, then $lg\frac{2a+3b}{5} = \frac{1}{2}(lga + lgb)$.

The problem can be solved through both consistent applications of analysis (under the scheme of Euclid or scheme of Pop) and synthesis, and by direct application of synthesis, together with the particular method – completing to a perfect square root and heuristic method "approaching" the given to the thesis of the problem.

Problem D2. Prove that, if $a = x^2 + xy + xz,$ $b = y^2 + xy + yz$ and $c = z^2 + xz + yz$, then $ayz + bxz + cxy = \frac{3abc}{a+b+c}$.

The problem is multifunctional, because, except for maintaining in the ZAD of the students their skills to use only synthesis, it serves to consolidate the skills to apply the method of identity transformations (particularly-mathematical method), and also skills to transfer problems by equivalent transformations of the conclusion until receiving a problem, the solution of which is getting easier.

Problem D3. It is given an angle POQ, equal to 60°. A point M is taken, which is internal to the angle and is located at distances $MA = a$ and $MB = b$ respectively from its sides OP and OQ. Find the length of the segment OM.

The solution of the problem can be searched by sequential or parallel applications of analysis and synthesis, as well as only by direct application of synthesis. At the same time it allows two ways to solve, corresponding to two additional constructions – the continuation of the segment AM to cross to the other side (Fig. 7); or construction the segment AB and a circle with a diameter OM (Fig. 8).

It is clear that in solving the problem there are used particularly-mathematical methods, such as "auxiliary circle", "auxiliary segment", algebraic method and others. On the base of the theory of parametrization[61] (Portev, 2001), many other problems can be created from this problem (by varying the given and/or the conclusion) in the solving of which the same methods are used.

[61] Portev, L. (2001) Definitness (parametrization) of geometrical figures and applications. In: V. Miloushev (Eds.), *Methods for solving problems* (*from the school course in mathematics*). Part I. (pp. 177-193). Plovdiv, Bulgaria: Makros. (p. 180). [In Bul.]

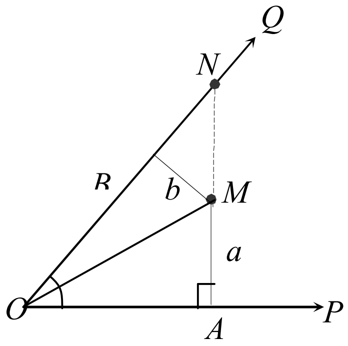


Figure 7.

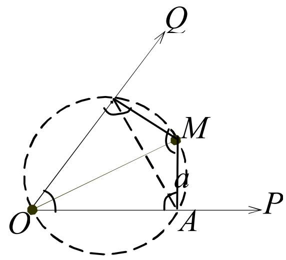


Figure 8.

Problem D4. In a triangular pyramid $SABC$ the lateral edge SA is equal to the base edge AB. The angle that SC concludes with the base of the pyramid is equal to 60°. The points A, B, C and the midpoints M, N and P respectively of the lateral edges lie on a sphere, the radius of which is equal to 1, $\angle BAC = 30°$. Find the volume of the pyramid.

A large amount of information is extracted by synthetic reasoning from the given of the problem, which requires the initially drawing (Fig. 9) to be improved continuously (see for example Fig. 10), which, in turn, eases considerably the searching of the solution (for example, by using and auxiliary Fig. 11). The problem allows a purely geometric solution, which is too interesting as well.

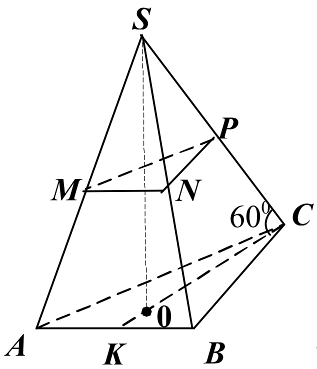


Figure 9.

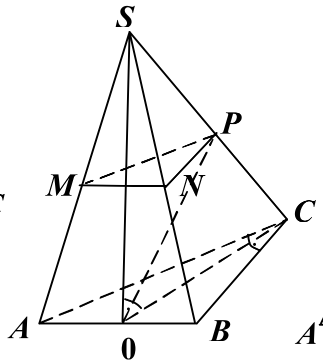


Figure 10.

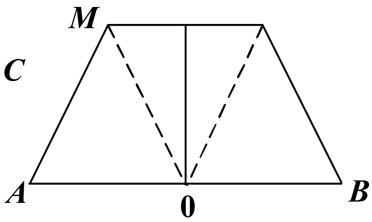


Figure 11.

Problem D5. Thete is given a rectangular triangle ABC ($\angle ACB = 90°$), for which $BC = 5$ and $BC > AC$. The height CD $(D \in AB)$ is drawn up. In the triangle BCD is inscribed circle with center O and radius equal to 1. Find the attitude of the areas of the triangles, in which the line AO divides triangleABC (Fig.12).

The finding a solution of the problem is carried out through multiple transitions from analysis to synthesis and vice versa. This determines the complicated structure of this solution. Experience shows that after successfully dealing with the problem, it delivers satisfaction and joy to the students, which increases their interest in mathematics.

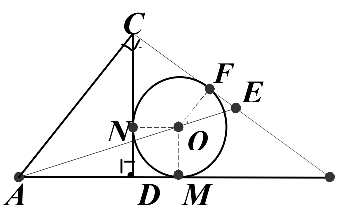


Figure 12.

Problem D6. Prove that if $ab + bc + ca = abc$, $a^3x = b^3y = c^3z$, $a \neq 0$, $b \neq 0$, c$\neq$ 0, then:

$\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} = \sqrt[3]{a^2x + b^2y + c^2z}$.

This problem is also multifunctional, since, except for maintaining in the ZAD of students their skills for parallel application of analysis and synthesis, it serves to consolidate the skills to apply the so called method of immediate check up[62] (Slavov, 1969), and also skills to use the analogy as heuristic method (see monograph[63] (Skafa & Miloushev, 2009)).

Problem D7. It is given a right triangular pyramid $ABCM$, in which MO is a height, and P is a midpoint of MO. (Fig. 13)

a) If the attitude of the lengths of a lateral edge and a basic edge is k, then express the cosine of the angle APB by k;

b) If $\angle APB = \varphi$, find the attitude k of the lengths of a lateral edge and a basic edge;

c) If $\angle APB = 90°$, prove that the pyramid $ABCM$ is a regular tetrahedron.

(The problem is a modified version of a competition problem for the universities in Bulgaria – 1985).

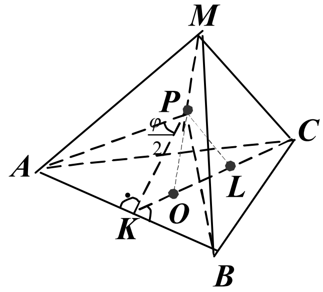


Figure 13.

The problem can be solved at least in four ways – two algebraic (one using the vector method) and two geometric. One of the geometric ways is based on the specific information in the problem and is extremely rational, accessible and "beautiful," but it is necessary an additional construction – drawing the mid-segment PL of the triangle MCO to be carried out.

Problem D8. A rectangular triangle ΔABC ($\angle ACB = 90°$) is given. If point O is the center of internally inscribed circle in the triangle, $AO = \sqrt{5}$ and $BO = \sqrt{10}$, find the area of the triangle ΔABC (Fig. 14).

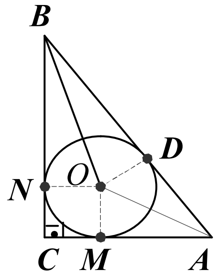


Figure 14.

The searching of a solution is accomplished using a joint application of analysis and synthesis (i.e. analysis through synthesis), on the base of which we reache to a basic problem for solving a triangle (therefore the method of its solving is known). We will note that in this case the method "analysis through synthesis" has a great heuristic value as reduces the solving of the original problem to a problem with a known method for solving that, in turn, includes the private-mathematical method of the "binding element"[64] (Lalchev & Vutova, 2003). The last step of the synthesis can lead to a second way to solve, namely, using the Pythagorean theorem, leads to irrational equation for the radius of the inscribed circle (which, however, is too complicated to solve).

Problem D9. It is given a triangle ABC, for which ($\angle BAC = \alpha = 45°$). The inner bisector AL ($L \in BC$) is drawn, and through point A is build a line p, perpendicular to AL, which crosses the line BC in point M, so that $BM = BA + AC$. Find the angles β and γ of the triangle ABC.

An important requirement to the solution of each problem is it to be *complete*. So even in the initial analysis of the text of the problem and its assimilation, with a view to make a believable drawing it must be noted that there are two options for crossing the line p (which is a bisector of the outer angle in the vertex A) with the line BC (Fig. 15 and Fig. 16).

The search of solution in both cases can be accomplished by joint or parallel application of analysis and synthesis (while the possibility of their joint application must be checked after each step). The special feature in the searching a solution to this problem (in any case) is that the subject, that solves the problem (pupil, student or

[62]Slavov, K. (1969). *Basic methods for solving problems in algebra*. Sofia, Bulgaria: Narodna prosveta. [In Bul.]

[63]Skafa, E. & Milloushev, V. (2009). *Constructing educational-cognitive heuristic activity in solving mathematical problems.* Plovdiv, Bulgaria: University press „Paisii Hilendarski". [In Bul.]

[64]Lalchev, Z. & Vutova, I. (2003). Binding element. In: *Mathematics and Mathematical Education,* (pp. 369-373). Sofia, Bulgaria: BAS. [In Bul.]

teacher) must, on one hand, performe, by analogy, a transfer of his knowledge and skills, related to the use of the very productive method of the "binding element" in conjunction with the algebraic method (creating a linear equation for the sought angle), and on the other hand – have to make such additional constructions that they give good opportunities to use the given in the problem, with the aim to realize these methods.

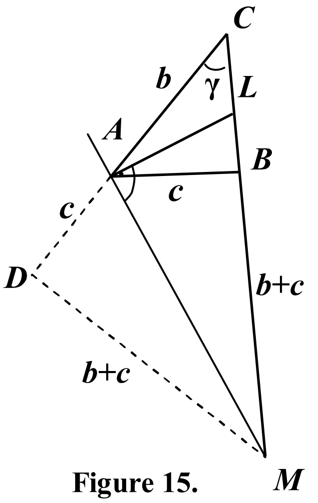


Figure 15.

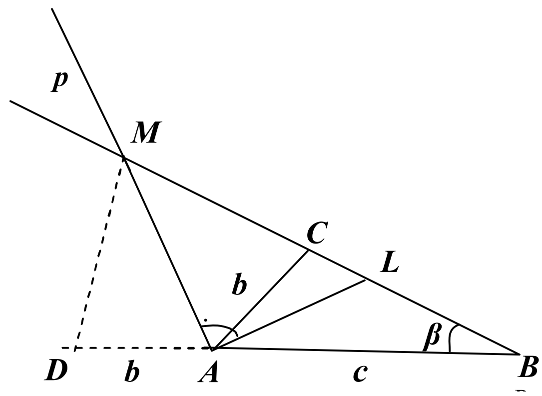


Figure 16.

The self-dealing with this problem is a sign, that the training of learners conducted in accordance with the praxeological reflection, which is constantly focused on acquring creative experience in applying various methods, including the method of the "binding element", as well as to improving their skills to use the generally-logical methods, has achieved the respective aims of the methodical system. We will emphasize that searching a solution through these methods especially of this problem, we have to express one and the same angle through one and the same parameter in two different ways, which (as we have discovered from our own practical experience working with prospective students) is a certain psychological difficulty for the learner – very often he falls in the so called "vicious circle".

In connection with improving the skills of students to meet the requirement of *completeness* of the solutions of mathematical problems, it is appropriate to discuss other problems with similar characteristics with them (eg. competition problems from §3 in the handbook[65] (Portev, Ivanov, Milloushev, Mavrova, Nikolov, Porteva, … & Pavlov, 2004)).

So far we have examined a number of options for mastering (in theoretical and practical aspects) the basic generally-logical methods – analysis, synthesis and some of their combinations. In the following exposition we will present methodological models for mastering some significant in didactic terms non basic generally-logical methods and several specific – particularly-mathematical methods for solving problems from the school course in mathematics. Their assimilation can be carried out in the process of their combined parallel application, and in many cases, together with the inclusion and maintenance in the zone of actual development (ZAR) of students their knowledge and skills to apply the basic generally-logical methods in solving problems of different types. These knowledge and skills of the students appear to be a base for an "introduction" and maintenance in their ZAD also knowledge and skills about the application of certain non-basic generally-logical and particularly-mathematical

[65]Portev, L., Ivanov, I., Milloushev, V., Mavrova, R., Nikolov, Y., Porteva, A., … & Pavlov, H. (2004). *Mathematics – a textbook for state graduation and entrance exam in four parts. Part II „Geometry".* Plovdiv, Bulgaria: Letera. [In Bul.]

methods for solving various mathematical problems, the theoretical base of which includes elements from logics and particle methods and algorithms established in algebra or geometry.

The models for mastering methods for solving problems in order to correspond completely to the idea of intellectual and praxeological reflection, they must comply with the following variety of the approach to learning *"students actively participate in rediscovery of scientific truths and acquire their own cognitive experience*"[66] (Georgieva, 2001).

We will note that in the previous part we didn't put the question about rediscovering basic generally-logical methods (synthesis, analysis and combinations of them), because students use them implicitly from the earliest grades, they evolutionary gain experience in the years and it is enough to build systematically knowledge and skills for them to a higher level by continuing "detachment" from specific carriers. Just in the upper classes, and it rarely, students have the opportunity to get acquainted with the method of negation, applied in certain problems and with the method of "logical algebra" – eventually in the compulsory education or in the optional courses. That's why we consider here especially the methodology of the activity of "rediscovery" knowledge about methods for solving problems (such as the method of the Logical algebra, the method of negation and etc., as well as some popular, practically significant particularly-mathematical methods) by learners.

The effectiveness of the methodology of the activity for acquiring knowledge and skills about non basic generally-logical and particularly-mathematical methods largely depends on the content and structure of the systems of educational mathematical problems through which the certain learning process is realized. In constructing appropriate didactic subsystems with problems, designed for "rediscovery" and mastery a particular method for solving and in the organization of the methodology of working with them, it is appropriate to be guided by the following considerations:

- the mastery of certain non-basic generally-logical and/or particularly-mathematical methods should be implemented jointly with the assimilation or consolidation of mathematical knowledge and skills for their application related with them, moreover with regular application of the "method of learning by summarizing reasoning", in order to provide opportunities for self-awareness of the methods used by learners. That's why the systems of problems must provide cyclic accumulation of relevant experience and generalization;
- for advanced students, after a certain stage of regular application of the "method of learning by summarizing reasoning" a summarizing activity at a higher level must be made. For example, in the stages "A look backwards" in solving a certain problem, it is appropriate to apply the so called *method of education through generalization, formalization and schematic modeling.* In the realization of this method the activity of the subject includes additional research of the general characteristics of the structure of the problems of the system and the related method for solving and also fixing this method in the mind of the students by its materialization through models;
- it is appropriate, in terms of the principles of structural completeness and accessibility, the problems of the system to be different in content, but with common elements of the cell of operator, while their invariance to be "accessible" to be distinguished from the students, and also to serve as a foundation for the formation and fixation of the appropriate method in their minds and reach to understanding of its essence;
- the searching and finding out solutions of the different problem-components must be carried out through basic generally-logical methods;

[66] Georgieva, M. (2001). *Reflection in mathematics education* (*V-VI class*). V. Turnovo, Bulgaria: Feber. (p. 17). [In Bul.]

- the problem-components of the relevant subsystems must be selected and structured in such a way, if possibly, that the methodology of working with them to appear to be a kind of a "mini-model" of the historical process of the emerging of the corresponding method for solving problems. It is well known that most of the methods have arisen as a result of natural adherence to the strategy that *the solving* of a given problem *must be reduced* to *considering another problem with a known method for solving* (based on a theory, that is already mastered by learners) or to *searching and discovering ways of solving a simple problem-components*. When this fact is revealed more often in mastering such methods, actually one important principle is realized, which can be called *"principle of permanency to the approach of the occurrence of a certain term in historical aspect"*;

- there must be provided an optimal number of problems (not only before, but also after applying the method of education by generalization, formalization and schematic modeling) with a view to possible correction of the formal or schematic models, as well as problems for individual work, with the aim to "insert" and maintain the certain method in the ZAD of learners to a level of self-awareness and self-improvement;

- the subsystems with problems, designed to master respectively non logically-general and particularly-mathematical methods must have a "amphitheatric" character.

We will stop briefly on the necessity of this last requirement and its nature. In the article of Grozdev & Kenderov (2005) an elaborate information about the basic characteristics and the process of realization of the European project "MATHEUS: identification, motivation and support of mathematical talents in European schools" is given. In the fourth part of this paper, called “The Idea of Amphitheatric Instruments”, on the base of some general laws of psychology, the authors explain and prove the idea about the so called systems of thematic mathematical problems of the “steps” type, which consist of ordered mathematical problems of increasing difficulty, so that each student should be motivated to reach the "marginal” step appropriate for his/her abilities (i.e. to reach the "local" self-actualization in the context of reflexive approach). One of these laws says that “...the full mastering occurs only in the process of **rediscovery”**[67] (Grozdev & Kenderov, 2005).

Conditions for the setting of this phenomenon, especially in acquaintance with a certain method and its mastering, may be provided successfully through activities, which are represented schematically on Fig. 17.

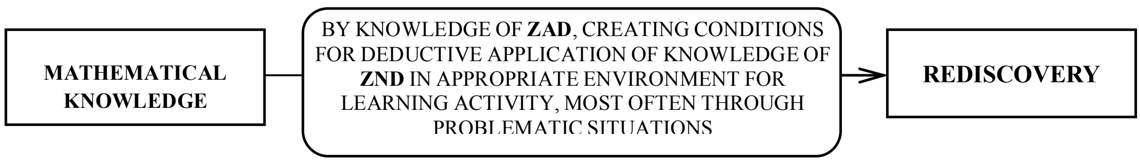


Figure 17. Model for accelerating the process of rediscovering a method

Our experience in this regard shows, that mastering a large part of non-basic generally-logical and particularly -mathematical methods for solving mathematical problems by students is carried out effectively through *rediscovering* according to this schematic model in the process of intense analytically-synthetic activity. Besides the developing effect, the rediscovery has also a motivating and practical importance – in solving a great part of mathematical problems. The methods that have been already mastered can be applied further together just with synthesis, which "*automatically" transforms the problems it is applied to into more simple ones.* This fact can be substantially used to motivate learners (pupils and students) to study the relevant methods.

[67]Grozdev, S. & Kenderov, P. (2005) Tools for finding out and supporting outstanding students in mathematics. In: *Mathematics and Mathematical Education,* (pp. 53-63). Sofia, Bulgaria: BAS. (p. 61). [In Bul.]

Mastering skills for application of some non-basic generally-logical methods and certain particular methods in solving mathematical problems by students is performed parallel with the consolidation of knowledge and skills for combined use of analysis and synthesis, using the didactic appropriate systems of problems of type "stairs" described above. In the following exposition we briefly present approbated systems of mathematical problems of this type, having the upper function, their formal descriptions (and hence generalized descriptions of the certain type of problems), and also formalizations of some of the methods.

For the sake of brevity further, each element of "extended definition" (definition or theorem-sign) of a mathematical concept, as well as any theorem-feature of such a concept, we accept provisionally to call "a theoretical unit." The theoretical basis of one mathematical problem consists of certain laws and rules of propositional and predicate logic (which essentially are also theoretical units) and of specific theoretical units of the field from which the problem is. As a result of their deductive application in a specific sequence, the solution of the problem is constructed.

We will demonstrate some utilized possibilities of specific theoretical units to generate methods for solving certain types of problems. These options are presented to students in a process of target actions for "rediscovery" and approbation of appropriate methods. The results show that this can promote effectively the recovery of the forming potential of the reflective-synergistic approach (particularly on the basis of cognitive (intellectual) reflection and praxeological reflection).

First we will discuss the issue of absorption of non-basic generally-logical methods for solving problems, by looking some problems of a subsystem, which is designed to absorb generally-logical methods, based on the rule "Logical algebar", including its particular case known in logic as "a rule of contraposition". For this purpose we will present two problems and their "training solutions" in the following comparative table (Tabl. 2).

Table 2. Training solutions of problem 1 and problem 2

Problem 1. Prove that if a three-digit number is not divisible by 37, then this number can not be recorded with the same numbers. **Solution:** We apply the rule of contraposition (T2) If a three-digit number can be written with the same numbers, this number is divisible by 37. Proof of the new problem: Use the statement (T1): Each a three-digit number $\overline{abc}$ can be written as $\overline{abc} = a \cdot 100 + b \cdot 10 + c$; We have: $\overline{aaa} = a \cdot 100 + a \cdot 10 + a$. Proving that this sum is divided by the number 37 is based on the already learned knowledge: - the property $ax + bx = (a + b)x$; - the statement "if in one product, one of the multipliers is divided into a given number, then the product is also divided into the same number." We received: $\overline{aaa} = a \cdot 100 + a \cdot 10 + a = 111a = 37 \cdot 3a$ It follows that the number $\overline{aaa}$ is divisible by 37.	***Problem* 2.** Prove that if $a^4 + b^4 < 2$, then $a + b \neq 2$. **Solution:** We apply the rule contraposition (T2) Prove that if $a + b = 2$, then $a^4 + b^4 \geq 2$ Proof of the new problem: T1: A system of appropriate formulas for short multiplication and method of substitution. Since $a + b$ is a symmetrical expression, then in the equality $a + b = 2$, can be substituted $\left\vert \begin{array}{l} a = 1 + k \\ b = 1 - k , k \in R \end{array} \right.$ The difference is formed $a^4 + b^4 - 2$ and it is converted, using the substitution and T1 $a^4 + b^4 - 2 = (1 + k)^4 + (1 - k)^4 - 2 =$ $= 1 + 4k + 6k^2 + 4k^3 + k^4 + 1 -$ $-4k + 6k^2 - 4k^3 + k^4 - 2 =$ $= 2k^4 + 12k^2 \geq 0$ Therefore $a^4 + b^4 - 2 \geq 0$, i.e. $a^4 + b^4 \geq 2$.

After solving the problems (with the help of the teacher possibly), it is appropriate to conduct a proper frontal talk with the aim to direct students to analyze the actions and means of their realization in the process of solving various problems and, on the base of comparison to "separate" the similarities and differences in the means they have used and to ask them make appropriate conclusions. In particular, the attention should be payed to the following three points:

• **first**, in each problem of the system the rule of contraposition (in short, provisionally we name it **"a system of statements M",** although in this case just one statement is used) served to create conditions for successful application of a statement and/or a system of statements (which is briefly named **"a system of statements T"**), namely:

- in *problem* 1. the system **T** consists of the statement: „each a three-digit number $\overline{abc}$ can be written as $\overline{abc} = a \cdot 100 + b \cdot 10 + c$ ”;

- in *problem* 2. the system **T** is formed by the formulas for short multiplication and method of substitution used in solving the problem;

• **second**, the system **M** leads to a standard problem, namely:

- in *problem* 1. it is „Prove that the sum $a \cdot 100 + a \cdot 10 + a$ is divided by the number 37”;

- in *problem* 2 – „Evaluate the sign of the difference $a^4 + b^4 - 2$”;

• **third**, the system **M** can can be applied to other, a relatively large number of problems, in order to lead them to problems of a familiar type.

The joint consideration of these problems can help for the "detachment" from the common in the way of their solution, namely the basic role of the rule of contraposition (to replace a given problem with an equivalent other problem, the theoretical basis of which is found easier). After considering other similar problems, a method for solving mathematical problems, the given and/or the conclusion of which contain "inconvenient" negatives is formed and fixed in the minds of students. This approach, demonstrated with the presented system of problems, can be considered in general as a model, similar in some respect to a (heuristic for its time) model for differentiation in the theory and practice of so called „method of contraposition" for solving problems.

To get the students to the idea of using the "Logical algebar"[68] (Ganchev, 1988) in its general case, we may offer them to consider the following two problems, to compare their given and their conclusions and to ask students to try to establish whether the problems are equivalent or not.

Problem 3. Prove that if, $a^2 + b^2 + c^2 = 1$, $a^3 + b^3 + c^3 = 1$ and $abc \neq 0$, then $a + b + c \neq 1$.

Problem 4. Prove that if $a^3 + b^3 + c^3 = 1$ and $abc \neq 0$, then $a + b + c \neq 1$ or $a^2 + b^2 + c^2 \neq 1$.

Applying the rule of contraposition and the law of de Morgan on problem 3, students will receive the following problem, which equivalent to the given one:

„If $a + b + c = 1$, then it is satisfied that $a^2 + b^2 + c^2 \neq 1$ or $a^3 + b^3 + c^3 \neq 1$, or $abc = 0$”

On one hand, it appears, that the resulting problem differs structurally from problem 4, and on the other hand – since the problem contains three statements in the given, two of which are negations, it appears to be more difficult for solving and more problematic than the original problem 3. The same conclusion is made, if apply similarly the rule of contraposition and the law of de Morgan to problem 4. Therefore, the application of the simplified version of the Logical algebar (the rule of contraposition) is not successful in the examples presented above. Thus the students can be motivated to master

[68]Ganchev, I. (1988). Logical models in the methodology of education in mathematics. In: *Mathematics and informatics*, Part II, (Proceedings of „Anniversary Conference 25 years of the University of Shumen ”), (pp. 49-52). Shumen, Bulgaria. [In Bul.]

the "Logical algebar" in its general case. After getting acquainted with this rule, the method, based on it can be rationalized on the base of the method of education by summarizing reasoning in the process of solving these problems and to be "approbated" in solving other similar problems.

We will note that after applying of the rule "Logical algebar" the following problem is obtained:

Problem E1. "Prove that, if $a + b + c = 1$, $a^2 + b^2 + c^2 = 1$ and $a^3 + b^3 + c^3 = = 1$, then $a.b.c = 0$", which is appropriate to be discussed and solved with students in advance. Then it's too possible the students to notice that there is a connection between this problem and the couple of problems (3 and 4) from the system above, and thanks to the acquaintance with the new rule they can assimilate what this relationship constitutes in (namely in the **equivalence**, based on the rule of "Logic algebar").

The solution of problem E1 also has a characteristic feature that it can be discussed in terms of generally-logical method "complete induction" for solving problem-components. We would like to describe here this comment with purely didactic reasons, that's why we present its solution.

Solution: Using the third given equality in problem E1 and the system of formula (**T**) the following consequences are received:

$a^3 + b^3 + c^3 = 1 \Rightarrow a^3 + b^3 = 1 - c^3 \Rightarrow$

$\Rightarrow (a + b)(a^2 - ab + b^2) = (1 - c)(1 + c + c^2)$ (1)

In addition, from the first given equality $a + b + c = 1$ follows, that $a + b = 1 - c,$ and from the second one follows $a^2 + b^2 = 1 - c^2$. Then the equality (1) consistently takes the form

$(1 - c)(1 - c^2 - ab) - (1 - c)(1 + c + c^2) = 0 \Rightarrow$

$\Rightarrow (1 - c)(1 - c^2 - ab - 1 - c - c^2) = 0 \Rightarrow$

$\Rightarrow (c - 1)(2c^2 + c + ab) = 0$ (2)

In order to express the monomial *ab* by the variable *c*, there can be used the given $a + b = 1 - c$ From it the equality $a^2 + b^2 + 2ab = 1 - 2c + c^2$ follows and from here and the given $a^2 + b^2 = 1 - c^2$ follows that $1 - c^2 + 2ab = 1 - 2c + c^2$, from where after some simplifications is found that $ab = c^2 - c$. Then the equality (2) takes the form $3c^2 \cdot (c - 1) = 0$ (3)

From equality (3) two possibilities are obtained: $c = 0$ or $c - 1 = 0$.

1. If $c = 0$, then it is obvious that $abc = 0$.
2. If $c - 1 = 0$, i.e. $c = 1$, then from the given $a^2 + b^2 + c^2 = 1$ is consistently obtained that $a^2 + b^2 + 1 = 1 \Rightarrow a^2 + b^2 = 0 \Rightarrow a = b = 0$. Therefore $abc = 0$.

This problem is solved, and since it is equivalent to both *problem* 3 and *problem* 4, they are also considered to be solved.

In order the students become able to make a transfer of the "methodology" for the formation of the method of the rule of contraposition for solving problems, it is appropriate to direct them to consider once again the reduction of problems 3. and 4. to the **problem E1**:

"Prove that, if $a + b + c = 1$, $a^2 + b^2 + c^2 = 1$ and $a^3 + b^3 + c^3 = 1$, then $abc = 0$".

They have to consider the solution of this problem, while noticing which statements play the role respectively to the system T and the system M and to verify if the requirements pointed above for the formation of new method for them are accomplished. In other words, they have to be directed, on the base of the formal description of the preceding method, to make an equivalent transformation and "new interpretation" of the solution of the resulting problem which they knew, in order to realize the essence of the method "Logical algebar" in general terms.

So, with the possible help from the teacher, the students may find out that the system of statements **T** consists now of formulas for short multiplication, which they know and the system **M** – only of the rule "Logical algebar". Using this, they can comment the transformation of problems 3 and 4, and also the way of their solving.

The solution of problem E1 is useful to be comment with students – future teachers in mathematics, in particular the solution of one problem-component, which requires the consideration of two cases about the possible values of number ***c***. This comment gives reasons to acknowledge the students with the method of complete induction, in more details, which method is commonly used in school in solving mathematical problems or their components. In order to introduce this method in the zone of near development (ZND), and after that in the zone of actual development (ZAD) of students, we recommend to construct a model for creating conditions for applying the method of full induction (**M1**) together with the students, in a cese when, for example, a system **T** of elements from the theory serves to break up the field of a given problem into equivalence classes such that the problem considering separately in each class, to be solved by known algorithm (**M2**). As a result of summarizing activity we have reached to the following conclusion: generally the inductive method is a common one, because it is applicable not only for all mathematical fields, but also for other scientific fields such as physics, chemistry, etc. But, when the full induction is applied successfully for solving a whole class of problems on the base of one and same elements from the mathematical theory, then it is possible to arise a ground for separation a corresponding particularly-mathematical method (though it is closely associated with the induction). For example such method is the so called method of intervals for solving equations, inequalities and systems, containing the unknown in module, fractional inequalities and etc.

After examining systems with educational mathematical problems, as the ones described above, some time should be taken to compare the methodology of work on mastering the different methods. Noting the similarity in this methodology, the university students can be asked to summarize, formalize and formulate a common approach for differentiation and fixation appropriate methods for solving mathematical problems in the minds of school students. Here we will briefly describe one formalization of these systems of problems carried out together with our students, in terms of constructing and structuring the systems, according to their purpose.

In constructing a system of mathematical problems the function of which is to consolidate knowledge of certain systems **T** and/or **M** of theoretical units and/or skills for application of methods assimilated by learners and at the same time – to master a particular newly introduced method (which, as a summarized operator includes knowledge about the systems **T** and **M**), it is necessary that the knowledge about **T** and **M** to be components of the cell of the operator of each problem of the system (i.e. **T** and **M** must appear to be invariants in the set of cells of operators of the problems of the system, while it's possible that **T** and/or **M** include in themselves logical laws or rules).

The university students suggested that, as a result of applying the method of learning by summarizing reasoning, learners should reach inductively to the conclusion that, with the help of knowledge about **T** and **M** and the skills to implement this knowledge, as well as the approach used for searching and finding out solutions of the problems of the system, many other problems outside the considered system can be solved as well. In this case it is very likely, a certain non-basic generally-logical method which is based on the knowledge about **T** and **M** to be separated and fixed in the minds of students with their active participation. It is better that this method is given a name. The students accepted, conditionally, to give the following name of the method: "a method of a particular component of the system with theoretical units **M**". At the same time they realized that in practice, usually the names of these methods are precised and specified according to the specifics of theoretical units that are used in them (e.g. method of the rule "Logical algebar", the inductive method of intervals and more.).

Now we will look at the problem of "rediscovery" by learners (pupils and students) particularly-mathematical methods for solving problems.

While non-basic generally-logical methods are based on certain laws of propositional and predicate logic (their study is not provided in the educational documentation), the particularly-mathematical methods, which are studied in the secondary school, are based mainly on planned in the curricula specific theoretical units from algebra and geometry. Due to this fact, as we noted above, the subsystems of teaching mathematical problems for mastering the individual particular methods must be constructed according to the relevant theoretical material. Now we are going to present examples of such subsystems that are suitable to be discussed with high school students.

Problem F1. Prove that, if $+b+c<2,\ a\geq -\frac{1}{2},\ b\geq -\frac{1}{2}$ and $c\geq -\frac{1}{2}$, then the inequality is satisfied $\sqrt{2a+1}+\sqrt{2b+1}+\sqrt{2c+1}<5$.

Usually, when students meet a problem with radical, they are trying to "get free" of it – by ranking in the corresponding exponent or by a suitable substitution. But in this problem there is a sum of three radicals and, obviously, the first idea is not relevant, and the second one leads to the emergence of three new variables and thas's why it is not suitable as well.

I way. *A sequential applying of analysis and synthesis*.

In conducting analysis, essentially, the learner asks several times the principal question: "In order to prove … , referring to the given condition (or any other reason) … what is it sufficient to prove before?".

In this problem, in order to prove that $\sqrt{2a+1}+\sqrt{2b+1}+\sqrt{2c+1}<5$, having in mind the given relationship $a+b+c<2$, it is sufficient to present the inequality in the following way:

$\sqrt{2a+1}+\sqrt{2b+1}+\sqrt{2c+1}<2+3$,

which, in its turn, considering the given relationship would be met, if it is proven the correctness of inequality:

$\sqrt{2a+1}+\sqrt{2b+1}+\sqrt{2c+1}<a+b+c+3$.

The experience gained from the transformation of algebraic expressions on relevant rational ways leads to "heuristic" idea that, in order to use the presence of "symmetry", the last inequality is appropriate to present in the following symmetric form:

$\sqrt{2a+1}+\sqrt{2b+1}+\sqrt{2c+1}<(a+1)+(b+1)+)c+1)$.

In order to satisfy this inequality, it is enough to apply the inequalities

$\sqrt{2a+1}\leq a+1;\ \sqrt{2b+1}\leq b+1;\ \sqrt{2c+1}\leq c+1$,

while at least one of them is strict.

In order these three inequalities to be true, it is enough to meet the following inequalities:

$2a+1\leq a^2+2a+1,\ 2b+1\leq b^2+2b+1$ and $2c+1\leq c^2+2c+1$,

i.e. $0\leq a^2,\ 0\leq b^2$ and $0\leq c^2$, which is obviously satisfied if at least one of the numbers *a, b* or *c* is not zero.

Synthesis. Let $a=b=c=0$. Then we have $\sqrt{2a+1}+\sqrt{2b+1}+\sqrt{2c+1}=1+$ $+1+1<5$.

Now let's at least one of the numbers *a, b* or *c* is not zero. Then $0\leq a^2$, $0\leq b^2$ and $0\leq c^2$, while at least one of these inequalities is strict. From them there are consistently received these inequalities:

$2a+1\leq a^2+2a+1,\ 2b+1\leq b^2+2b+1$ and $2c+1\leq c^2+2c+1$,

$2a+1\leq (a+1)^2,\ 2b+1\leq (b+1)^2$ and $2c+1\leq (c+1)^2$,

$\sqrt{2a+1}\leq |a+1|,\ \sqrt{2b+1}\leq |b+1|$ and $\sqrt{2c+1}\leq |c+1|$,

$\sqrt{2a+1}\leq a+1,\ \sqrt{2b+1}\leq b+1$ and $\sqrt{2c+1}\leq c+1$.

Since at least one of these three inequalities is strict, as a result of summing their relevant sides, follows the inequality $\sqrt{2a+1}+\sqrt{2b+1}+\sqrt{2c+1}<a+b+c+3$, and using the given condition $a+b+c<2$, final is obtained the inequality:

$\sqrt{2a+1}+\sqrt{2b+1}+\sqrt{2c+1}<5$.

The problem is solved.

In the stage "A look backwards" the following didactic task is assigned: "To determine *the key step* in the synthetic solution and on its base to solve the problem applying just analysis, (i.e. without the help of synthesis)".

The learners noticed, that the key step in the solution is receiving the inequalities

$2a+1\le a^2+2a+1,\ 2b+1\le b^2+2b+1$ and $2c+1\le c^2+2c+1$.

In their motivation there is used the term "breaking" of equality through appropriate "unbalancing variable", which in purely theoretical terms, is a form of verbalization, for example, of the combined application of the transitive property of the relations "equal" and "less" or "more", as well as the disjunction of them, on one hand, and monotonous property of the operation addition, on the other hand $(x<y\ \Rightarrow x+z<y+z)$.

For example, from $y=y$ and $0\le z^2$ follows the inequality $y+0\le y+z^2$, and from the equality $x=y$ and the inequality $y\le y+z^2$ follows, that $x\le y+z^2$.

Thus, using this idea, a new way to solve the problem is differentiated on the basis of the reasoning given above which the learners realize as follows:

II way. Obviously, when $a=b=c=0$, as above, the following inequality is satisfied $\sqrt{2a+1}+\sqrt{2b+1}+\sqrt{2c+1}=1+1+1<5$.

In the case, when $a\ge-\frac{1}{2}$, using the equality $2a+1=2a+1$ and the inequality $0\le a^2$, the following inequality is received $2a+1\le a^2+2a+1$. On the base of it and the fact, that the irrational function $y=\sqrt{x}$ is monotonically increasing in its defined area [0; +∞), the following inequalities are consistently received.

$\sqrt{2a+1}\le\sqrt{a^2+2a+1}\ \Rightarrow\ \sqrt{2a+1}\le\sqrt{(a+1)^2}\ \Rightarrow\ \sqrt{2a+1}\le|a+1|\ \Rightarrow$ $\sqrt{2a+1}\le a+1$.

Analogically, when $b\ge-\frac{1}{2}$ and $c\ge-\frac{1}{2}$ it is led to the inequalities:

$\sqrt{2b+1}\le b+1$ and $\sqrt{2c+1}\le c+1$.

From the last three irrational inequalities through summing their relevant sides is obtained:

$\sqrt{2a+1}+\sqrt{2b+1}+\sqrt{2c+1}\le a+b+c+3$,

and when we use also the given condition $a+b+c<2$, finally follows the result:

$\sqrt{2a+1}+\sqrt{2b+1}+\sqrt{2c+1}<5$.

In a similar way the following problem is seen.

Problem F2. Prove that, if $a+b+c+d\ge1$, $a\le\frac{1}{2}$, $b\le\frac{1}{2}$, $c\le\frac{1}{2}$ and $d\le\frac{1}{2}$, then the inequality is satisfied $\sqrt{1-2a}+\sqrt{1-2b}+\sqrt{1-2c}+\sqrt{1-2d}<3$.

Here we will directly present the synthetic solution "rediscovered" by students.

Since $a\le\frac{1}{2}$, then $1-2a\ge0$. Using the equality $1-2a=1-2a$ and the inequality $0\le a^2$, the following inequality is received $1-2a\le1-2a+a^2$. On the base of it and the fact, that the irrational function $y=\sqrt{x}$ is monotonically increasing in its defined area [0; +∞), the inequality $\sqrt{1-2a}\le1-a$ is obtained at the end.

Analogically, when $\le\frac{1}{2}$, $c\le\frac{1}{2}$ and $d\le\frac{1}{2}$ it is led to the inequalities

$\sqrt{1-2b}\le1-b$, $\sqrt{1-2c}\le1-c$ and $\sqrt{1-2d}\le1-d$.

Since, due to the condition $a+b+c+d\ge1$, the numbers a,b,c,d cannot be at the same time equal to zero, then at least one of the inequalities $0\le a^2$, $0\le b^2$, $0\le c^2$ or $0\le d^2$ is strict. Then, through summing the relevant sides of these four irrational inequalities is obtained:

$\sqrt{1-2a}+\sqrt{1-2b}+\sqrt{1-2c}+\sqrt{1-2d}<4-(a+b+c+d)$,
and using and given condition $a+b+c+d\geq 1$, final the inequality:
$\sqrt{1-2a}+\sqrt{1-2b}+\sqrt{1-2c}+\sqrt{1-2d}<3$ is obtained.

The following *didactic task* is given to the students – to compare the synthetic solutions of both problems and in particular, to compare the key elements in them, to distinguish what is common between these elements and to think about if it can be taken as a basis approach for solving similar problems (without conducting analysis). As a result of the performance of this task the learners concluded that in both cases an "increasing" of one side of the equality is done until receiving a suitable for the certain case inequality. In order to develop a reflexive thinking in the students, they are offered to describe briefly this activity in general case. This leads to the following brief description: **"enhancement" of an algebraic sum to obtain a perfect square.** Further the following question is discussed: in what cases this approach for solving problems is appropriate to be use? It was found that since the ideas of "grading" or "substitution" do not work, then a way for the release of radical through **"enhancement"** can be tried with a view to complete to a perfect square.

In this way learners enreach their knowledge about methods for searching and finding proofs of irrational inequalities (such as grading into an appropriate degree, using an appropriate substitution) with the new method **"enhancement"** of an algebraic sum to obtain a perfect square.

Thus, in practice, a system of methods for solving problems of any type can be consistently built.

In order to introduce the newly adopted method in the ZAD, the students can be divided into two groups. One of them can be asked to do (a group) generalization of Problem F1, and the other group – of Problem F2, after that they can be asked to discuss collectively the results that they have received. As a result of conducting a group work is led to creating and solving the following problems using the innovative method.

A generalization of Problem F1. Prove that if for any integer number $n\geq 1$ the inequalities are met $a_1+a_2+\cdots+a_n<2$, $a_1\geq -\frac{1}{2}$, $a_2\geq -\frac{1}{2}$, …, $a_n\geq -\frac{1}{2}$, then the following inequality is also met $\sqrt{2a_1+1}+\sqrt{2a_2+1}+\cdots+\sqrt{2a_n+1}<2+n$.

Now learners are even more convinced that to release from the radicals, the problem cannot be "attacked" by raising each side to a certain grade or with a suitable substitution. (In this case there is a sum of ***n*** radicals, that's why the first idea is not relevant, and the second one leads to the emergence of new *n* variables and is also inappropriate.) They share that the experience by examining the previous two problems leads them relatively quick to the following synthetic reasoning.

Solution. If $a_1=a_2=\cdots=a_n=0$, then it is obviously fulfilled that:
$\sqrt{2a_1+1}+\sqrt{2a_2+1}+\cdots+\sqrt{2a_n+1}=1+1+\cdots+1=n<2+n$
and therefore the inequality that must be proved is fulfilled.

Now let's at least one of the numbers $a_1, a_2, \dots, a_n$ is not zero. Using the method **"enhancement"** of an algebraic sum to obtain a perfect square the following inequalities are received:

$2a_1+1\leq a_1^2+2a_1+1,\ 2a_2+1\leq a_2^2+2a_2+1,\ \dots,\ 2a_n+1\leq a_n^2+2a_n+1.$

On the base of the fact, that the irrational function $y=\sqrt{x}$ is monotonically increasing in its defined area [0; +∞), then from the last inequalities follow consequently the inequalities:

$\sqrt{2a_1+1}\leq\sqrt{a_1^2+2a_1+1}\Rightarrow\sqrt{2a_1+1}\leq\sqrt{(a_1+1)^2}\Rightarrow$
$\Rightarrow\sqrt{2a_1+1}\leq|a_1+1|\Rightarrow\sqrt{2a_1+1}\leq a_1+1,$
$\sqrt{2a_2+1}\leq\sqrt{a_2^2+2a_2+1}\Rightarrow\sqrt{2a_2+1}\leq\sqrt{(a_2+1)^2}\Rightarrow$
$\Rightarrow\sqrt{2a_2+1}\leq|a_2+1|\Rightarrow\sqrt{2a_2+1}\leq a_2+1,$

. . . .,

$\sqrt{2a_n+1} \leq \sqrt{a_n^2+2a_n+1} \Rightarrow \sqrt{2a_n+1} \leq \sqrt{(a_n+1)^2} \Rightarrow$

$\Rightarrow \sqrt{2a_n+1} \leq |a_n+1| \Rightarrow \sqrt{2a_n+1} \leq a_n+1,$

where at least one of them is strict.

So, we have proved the inequalities:

$\sqrt{2a_1+1} \leq a_1+1, \sqrt{2a_2+1} \leq a_2+1, \dots, \sqrt{2a_n+1} \leq a_n+1.$

From these inequalities through summing their relevant sides the following inequality is obtained:

$\sqrt{2a_1+1}+\sqrt{2a_2+1}+\cdots+\sqrt{2a_n+1} \leq a_1+a_2+\dots+a_n+1+1+\cdots+1,$

$\sqrt{2a_1+1}+\sqrt{2a_2+1}+\cdots+\sqrt{2a_n+1} \leq a_1+a_2+\dots+a_n+n,$

and using the given condition $a_1+a_2+\dots+a_n<2$, it is reached to the inequality $\sqrt{2a_1+1}+\sqrt{2a_2+1}+\cdots+\sqrt{2a_n+1}<2+n$, which we have to prove.

In a similar way, the following problem is created and discussed.

A generalization of Problem F2. Prove that if for any integer number $n \geq 1$ the inequalities are met $a_1+a_2+\cdots+a_n \geq 1$, $a_1 \leq \frac{1}{2}, a_2 \leq \frac{1}{2}, \dots, a_n \leq \frac{1}{2}$, then the following inequality is also met:

$\sqrt{1-2a_1}+\sqrt{1-2a_2}+\cdots+\sqrt{1-2a_n}<n-1.$

After solving and brainstorming the generalized problems, the students concluded that the ways applied for their solving can be developed into a particular method, which can be called briefly **"method for enhancement of a sum"** and it can be added to the other methods for proving irrational inequalities. As a result of its absorption, this particular method is included in *"the cell of operator"* of the problems of such a type. Later the method of enhancement of a sum could be further updated and used in solving appropriate problems of inequalities and even in the process of applying the synthetic method, as well as a method for solving relevant problems-components in the process of different versions of application of analysis and synthesis.

It should be noted that in such approach of education better conditions are created for effective implementation of the method of learning by summarizing reasoning and therefore it can be successfully used in work with advanced students. For this purpose, in the methodical work with students on the basis of an adequate frontal talk and some help from the teacher, which is aimed at the application of the method of education by generalization, formalization and schematic modeling, learners together with the tutor (***formalizing*** the activity of solving a certain type of problems) can reach to the following more general **model for rediscovery particular methods** for solving mathematical problems[69] (Milloushev, Frenkev & Millousheva-Boikina, 2005), which they can use further in the process of application the synthetic method (Fig. 18).

We will note that the accumulation of experience in the construction of this model allows the subject (trainee, trainer) to make the following **conclusion**: The particular method shortens the way of searching and finding out solutions of mathematical problems of the certain type, because in most problems it ignores the necessity of analysis. But for this purpose, the subject that solves the problem (pupil, student, teacher) must rediscover the method and must gain the necessary experience.

However, to realize that, first he must carry out consciously an analysis on a typical representative of a whole group of problems and to distinguish the cell of operator (the key element) which usually cannot be seen in synthesis by an inexperienced subject at first dealing with a problem of such a type. After mastering the method on a level of

[69]Milloushev, V. B., Frenkev, D. G. & Millousheva-Boikina, D. V. (2005). Model for teacning in rediscovery of particular methods for mathematical problem solving. *Proceedings of III Congress of Mathematicians of Macedonia*, Struga, 29.IX-2.X.2005, (pp. 123-130).

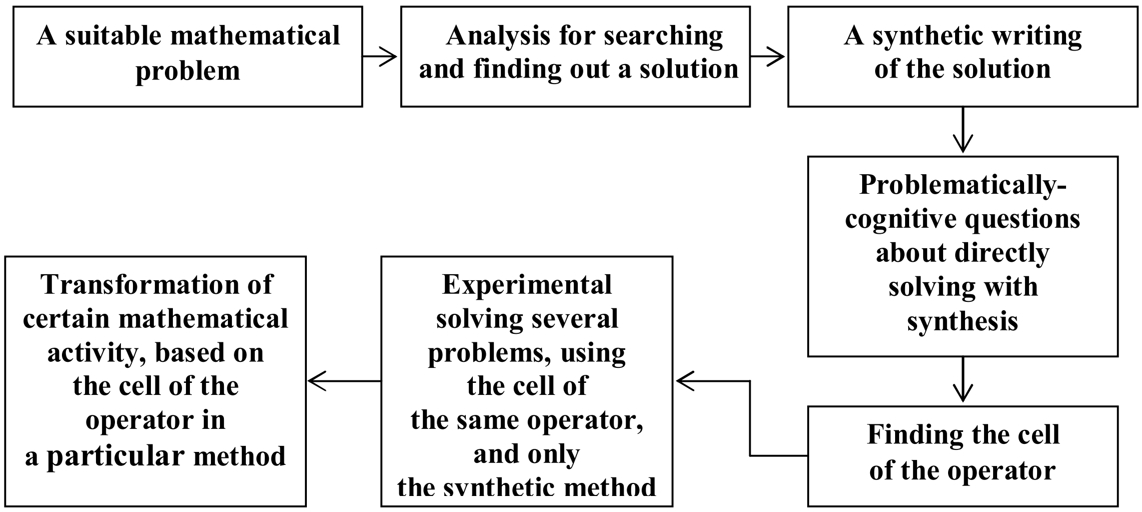


Figure 18. Model of the content and structure of the activity of rediscovery of particular methods for solving problems

direct application in the process of synthetic reasoning, in order to use it together with other methods, the subject should recognize (identify) the type of the problem and/or its sub-problem.

Other models for rediscovering methods to solve problems can be developed in collaboration with learners. After mastering each of the methods, the teacher should put students the question: "What in particular comprises the applicability of that method?". After some time the students will be able to "detach" the common thing of each method of its particular meaning, namely that each of the methods is based on a certain theory, which in its turn creates conditions for deductive application of another theory. (Thus for example, in the problems given above, the method of **"enhancement"** of algebraic sum to obtain a perfect square creates conditions for the implementation of the monotonous property of the sum in summing the relevant sides of one-way inequalities and for the performance of identical transformations in one and/or both sides of the resulting inequalities, in order to achieve the purpose of the problem).

In the next stages of work with systems of mathematical problems, designed to master certain methods for solving, it is appropriate to put as a research problem the requirement to be performed the treated hypothesis about the applied nature of the methods – the object of study – whether it is more or less credible. In this connection two aims should be set:

1) to "subjectify" the systems of problems towards their correspondence with the basic requirements for constructing such systems problems formulated at the beginning of the paragraph;

2) to "objectify" the conscious fundamental requirements through maximum active participation in the construction of systems of problems for mastering other partical methods of solving problems. (Of course, it is reasonable to discuss these methods that have been validated in the academic practice and have a great practical significance).

In realizing these aims the students might notice, that this non-basic generally-logical and some particularly-mathematical methods that have been discussed are mastered together with the assimilation or consolidation of knowledge connected with them and the skills for their implementation. However, most methods reduce the solving of a certain problem to considering another problem with a known method for solving or to searching and finding out ways to solve simpler problem-components. This means that while constructing systems with problems another important requirement must be respected: the problem-components of the system must be selected and structured in such a way, that the methodology of working with this system in some sense have to "copy" the historical process of the emergence of the corresponding method for solving problems.

In connection with the realization of the second aim – the "objectification" of the existing methodological knowledge, the following didactic task can be placed: to select a particular method and to construct a system of teaching mathematical problems for its absorption. Such, for example, could be the method of "an auxiliary angle" for solving trigonometric equations, which is popular in the school course of mathematics. For its full mastery by the students, on the base of the method of learning by summarizing reasoning, the teacher may select appropriate problems from the theme "Trigonometric equations" and differentiated them into a system. Here we will present the first three problems of such a suitable system, along with their solutions, and will briefly describe their general characteristics which should be "detached" from them, and to be separated and fixed in the mind of students.

1) $\frac{1}{2}sin x+\frac{\sqrt{3}}{2}cos x=\frac{1}{2}$;

Solution:

$sin\frac{\pi}{6}sin x+cos\frac{\pi}{6}cos x=\frac{1}{2}$;

$cos\left(x-\frac{\pi}{6}\right)=\frac{1}{2}$;

$\left\{\begin{array}{l} x-\frac{\pi}{6}=\frac{\pi}{3}+2k\pi \\ x-\frac{\pi}{6}=-\frac{\pi}{3}+2k\pi \end{array}\right.$;

$\left\{\begin{array}{l} x=\frac{\pi}{2}+2k\pi \\ x=-\frac{\pi}{6}+2k\pi \end{array}\right.$.

2) $sin x-cos x=-1$;

Solution:

$\frac{1}{\sqrt{2}}sin x-\frac{1}{\sqrt{2}}cos x=-\frac{1}{\sqrt{2}}$;

$\frac{\sqrt{2}}{2}\sin x-\frac{\sqrt{2}}{2}\cos x=-\frac{\sqrt{2}}{2}$;

$cos\frac{\pi}{4}sin x-sin\frac{\pi}{4}cos x=-\frac{\sqrt{2}}{2}$

$sin\left(x-\frac{\pi}{4}\right)=-\frac{\sqrt{2}}{2}$

$\left\{\begin{array}{l} x-\frac{\pi}{4}=-\frac{\pi}{4}+2k\pi \\ x-\frac{\pi}{4}=\frac{5\pi}{4}+2k\pi \end{array}\right.$;

$\left\{\begin{array}{l} x=2k\pi \\ x=\frac{3\pi}{2} \end{array}\right.$.

3) $\sqrt{3}\,sin x-cos x=\sqrt{2}$.

Solution:

$\frac{\sqrt{3}}{2}sin x-\frac{1}{2}cos x=\frac{\sqrt{2}}{2}$

$sin\left(x-\frac{\pi}{6}\right)=\frac{\sqrt{2}}{2}$

$\left\{\begin{array}{l} x-\frac{\pi}{6}=\frac{\pi}{4}+2k\pi \\ x-\frac{\pi}{6}=\frac{3\pi}{4}+2k\pi \end{array}\right.$;

$\left\{\begin{array}{l} x=\frac{5\pi}{12}+2k\pi \\ x=\frac{11\pi}{12}+2k\pi \end{array}\right.$.

Using the experience, based on analysis, synthesis, summary, the university students, and also a large part of the pupils noticed, that the following formulas are significantly used here:

$sin\, x\cdot cos\,\varphi \pm cos\, x\cdot sin\,\varphi = sin(x\pm\varphi)$, $cos\, x\cdot cos\,\varphi \pm sin\, x\cdot sin\,\varphi =$
$= cos(x\mp\varphi)$.

Then, as a system let's formally denote them with **T**. In order to apply them, however, the system of statements: $sin\,\varphi^2+cos\,\varphi^2=1$, $|sin\,\varphi|\le 1$, $|cos\,\varphi|\le 1$, should be substantially used to reach to the conclusion, that for any integer $p\le 1$ and for any integer число $q\le 1$ for which the equality $p^2+q^2=1$, is met, it can be found such an angle φ, that the equations $sin\varphi=p$ and $cos\varphi=q$ have to be met. Let's denote the system with all these "theoretical units" with **M**.

"Detaching" the common in the characteristics of the problems and their solutions (in the capacity as their particular meaning) the focus can be put on the following three points:

• on one hand, the system of statements **M** serves to create conditions for the application of the formulas in the system **T**;

• on the other hand, if using the canonical common type $a.sin\,x+b.cos\,x=c$ of the considered equations, by the system of statements **M** can be proved that there is a corner φ such that:

$$\frac{a}{\sqrt{a^2+b^2}} = cos\,\varphi \text{ and } \frac{b}{\sqrt{a^2+b^2}} = sin\,\varphi;$$

• from the last it is clear that **T** and **M** are also applicable to other, relatively large numbers of trigonometric equations (the formalization of which leads to their general description of the type $a \cdot sin\,x + b \cdot cos\,x = c$, where the coefficients a, b and c satisfy the inequality $c^2 \leq a^2 + b^2$).

All this leads to the separation and fixation in the mind of the learning subject the so called **"method of an auxiliary angle"** for solving trigonometric equations of the type $a \cdot \sin x + b \cdot \cos x = c$, under the specified conditions about a, b and c.

We will note that the formalization of this system of problems (in terms of their construction, as intended) can be accomplished analogically. So, after detecting the systems of theoretical units **T** and **M** and their role in the process of solving the problems of the system, it is found that as a result of the implementation of the components of **T**, the solving of a given problem is limited to solving a problem of known type (with a known method for solving). Moreover, a new moment occurs, namely: in the process of application the components of **T** a key role plays also the fact that from the components of **M** follows a particular (private affirmative) statement on the type "there exist an object ***p*** such, that it meets the equalities (connections) ... ", while these equalities also serve to reduce the solution of the given problem to solving a problem of known type.

By examining, analyzing and constructing other systems of suitable problems new applications of the discussed method for solving trigonometric equations can be revealed, namely, that the method of the auxiliary angle can be applied with the same success to solve more complex equations, having the type $f(a \cdot sin\,x + b \cdot cos\,x) = 0$, trigonometric inequations, for example $f(a \cdot sin\,x + b \cdot cos\,x) < 0$, as well to transform trigonometric expressions, which contains expressions of the type $a \cdot sin\,x + b \cdot cos\,x$. All this can be used to form skills in students for applying the method of the auxiliary angle in solving any problem that contains expression of the following type: $a \cdot sin\,x + b \cdot cos\,x$.

On the next stage of education the students may be required to apply the same methodology for rediscovery other methods that use "auxiliary elements". Analyzing a number of particilarly-mathematical methods for solving problems, it is noticed, that there are some methods which are also based on theoretical units, the role of which in the process of solving relevant problems is similar to the role of the system of theoretical units **M**, reflected in the summary of the previous type of systems with problems. Thus, for the teaching practice of students interest is the method of the "auxiliary circle". It is therefore appropriate to focus their thoughts in the following way. The approach that ishas been developed above for "rediscovering" methods for solving problems and their approbation in the process of working with specially designed for the purpose systems with educational mathematical problems refers, basically, to solving problems by means of algebra. We will note that this approach is in line with the trend of continuously incorporating elements of cognitive and praxeological reflection in the various methodological developments. At the same time the ideas contained there are common to many types of mathematical problems, it is therefore appropriate that this approach and the method which is "rediscovered" on its base, to be generalized, then specified and adapted (hence transferred) to solve problems by means of other mathematical fields. This leads to the notion that, at all, in solving problems of any mathematical field very often are used methods, an essential role in which plays the construction of a further model of the same area, which, in its turn, creates conditions for deductive application of certain theoretical units of the same area, leading to successfully solving the problems. Thus, in a working plan, the idea arises, that this method provisionally to be called **"method of an auxiliary model"**.

Therefore, the method of the "auxiliary angle" can be seen as a special case, and the approach of its "rediscovery" to be seen as a special case of more general approach, on the base of which learners could be introduced in general to any particular case of the method of the "auxiliary model". Now the method of the "auxiliary circle" which is interesting to the teaching practice in geometry (and is dealt with in this research), really appears to be one concretization of the method of the "auxiliary model", and it is at the same time an analogue of the method of "auxiliary angle" in trigonometry. For the students – future teachers in mathematics it is already clear the approach for constructing a system with problems, designed to mastering the method of the "auxiliary circle" by the school students. Here, as a model, we will present the following problem of such a system and its methodological development.

Problem G1. "The trapezium $ABCD$ is inscribed in a circle ($AB \parallel DC$). If the lines AD and BC intersect at point M, and the tangents built to the circle at the points B and D, intersect at point P, then prove that $MP \parallel AB$." (The problem is given on a regional math competition – II Correspondence Circle, "Iskra" Newspaper, Plovdiv, Bulgaria, March, 1982.)

Analysis. In order to prove, that $MP \parallel AB$ (Fig. 19), it is sufficient to find out that when we intersect them by a third line, equal corresponding angles are received or equal cross angles, or the sum of the two adjacent angles is 180°.

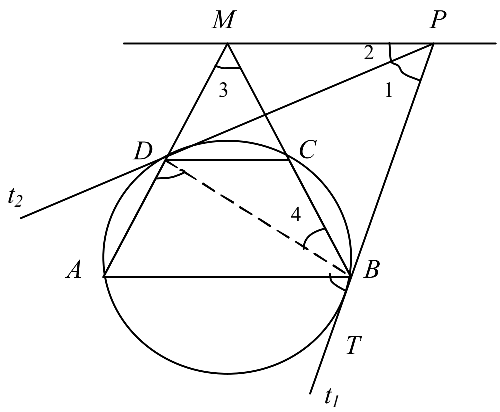


Figure 19.

An important moment in the choice of strategy for the problem solving is which of the lines, intersecting lines MP and AB, to be chosen as a “third” line. It is advisable to turn to those lines on which rays of angles lay which have a connection with the given circle and can be expressed with well known formulae. One of these suitable lines is for example the line BP, because the angles which it joins with MP and AB, have a direct connection with the circle. Therefore, in order to prove that $MP \parallel AB$, it is sufficient to prove that $\angle MPB = \angle ABT$. For the proof of this equality, it is sufficient to prove that $\angle ABT = \angle 1 + \angle 2$, where $\angle 1$ is a short symbol of $\angle DPB$, i.e. $\angle 1 = \angle DPB$, and $\angle 2 = \angle DPM$. Since $\angle ABT$ is a peripheral angle for the circle and it is measured with the help of the arc $\widehat{AB}$, and with the same arc is also measured the inscribed angle $\angle ADB$ (the diagonal DB is constructed), then in order to prove that $\angle ABT = \angle 1 + \angle 2$ it is sufficient to prove that $\angle ADB = \angle 1 + \angle 2$. But $\angle ADB$ is an exterior angle for the ΔDBM and then $\angle ADB = \angle 3 + \angle 4$, where $\angle 3 = \angle DMB$, and $\angle 4 = \angle DBM$. Therefore, in order to prove that $\angle ADB = \angle 1 + \angle 2$, it is sufficient to prove that $\angle 1 + \angle 2 = \angle 3 +$ $+\angle 4$. We examine two by two angles-addends in order to establish what connections exist between them. We notice that the rays of $\angle 1$ and $\angle 3$ go through the ends of the segment DB, and the rays of $\angle 2$ and $\angle 4$ – through the ends of segments DM. Therefore, in order that the equality $\angle 1 + \angle 2 = \angle 3 + \angle 4$, be fulfilled, it is sufficient to prove that the equalities $\angle 1 = \angle 3$ и $\angle 2 = \angle 4$, are fulfilled. As for the latter, it is sufficient to prove that around the tetragon $DBPM$ м it is possible to describe a circle. For the latter, it is sufficient to prove that either $\angle 1 = \angle 3$ or $\angle 2 = \angle 4$.

In connection with the application of the method of learning by summarizing and formalizing (intended primarily for university students), we will note, that in this problem, as well as in any other problem of the system (as specific carriers of the general characteristics of the ways for solving) the following two categories of elements of its theoretical basis can be distinguished:

a) The theoretical units, on which the sufficient conditions of the conducted analysis are based in order to reach a necessity to describe a circle around the rectangle $DBPM$ (conditions for parallel lines, properties inscribed and peripheral angle, the property of an outer angle of a triangle, etc.), and which at the same time will be used to obtain the necessary conditions in the synthetic representation of the solution. We can join them together and examine them in a system that, conventionally, denote with **T**.

b) The theoretical units, on which the proof of the equality $\angle 1 = \angle 3$ (or $\angle 2 = \angle 4$) is based, which, in turn, serves to implement a property for an inscribed quadrilateral, together with this property also unite into one system that denote with M.

It is clear, that the function of the system **M** here is also to create conditions for the application of theoretical units of the system **T**, with which, according to the analysis that is made, can be reached to successful solution of the problem.

Synthesis. Because $\angle 1$ is with an vertex outside of the circle, then

$$\angle 1 = \frac{\overset{\frown}{DAB} - \overset{\frown}{DCB}}{2} = \frac{\overset{\frown}{DA} + \overset{\frown}{AB} - \overset{\frown}{DC} - \overset{\frown}{CB}}{2} = \frac{\overset{\frown}{AB} - \overset{\frown}{DC}}{2}.$$

From the other side, $\angle 3 = \frac{\overset{\frown}{AB} - \overset{\frown}{DC}}{2}$.

Therefore $\angle 1 = \angle 3$, from where follows that a circle could be really described around the tetragon $DBPM$.

For the sake of brevity of the presentation, we shall not describe further the synthetic solution – after the construction of the newly found auxiliary circle around the tetragon $DBPM$, the solution becomes “transparent”.

From a methodological point of view it is appropriate when working with pupils (and also with university students) using the talk in the stage "A look backwards" to comment on the issue: “Is it possible to find out the way for the problem solving without such a detailed analysis?”. Our experience, while working with pupils and students shows that they usually answer: “Yes, but if through analysis we have reached the necessity of constructing the diagonal DB of the trapezium and we have already had enough practice to find out the existence of an **auxiliary** circle, on the base of the well known didactic system of indications for the description of a circle around a polygon”.

This didactic system of indications, each of which could appear as a cell of an operator-element of the system **M**, is the following:

Tl. If in the plane of a polygon there exists a point O, which is at equal distances from all its apexes, then a circle could be described around the polygon with center – point O.

T2. If the midperpendicuiars of all sides of one polygon cross in one point, then a circle could be described around the polygon with a center at this point.

T3. If the sum of two opposite angles of one tetragon is 180°, then a circle could be described around it.

T4. If one side of a polygon is seen from all other apexes under equal angles, then a circle could be described around the polygon.

In order to confirm students' knowledge and skills for this heuristic approach (finding out an auxiliary circle) as a method for problem solving of a larger class of problems, it is appropriate four types of systems of problems, separated as sub-systems to be created and suggested periodically to students (for provement, calculation, construction, problems from competitive exams and Olympiads), in the solving of which a more brief analysis is used and **“the additional construction – an auxiliary circle”** is effectively applied. When creating such didactic sub-systems with problems it is necessary to meet the requirement that the problems have to correspond to the principle of structural completeness concerning the complexity and problematicity of the problems in the system, and they also have to be subordinated to the idea, mentioned above, about the “amphitheatrical character of instruments for finding out and developing talents”.

Our practical experience shows that by pedagogical considerations is particularly useful to offer students – future teachers in mathematics, as a didactic task, to try to show empathy, putting themselves in the place of pupils, and from their point of view to mark the positive and the negative sides of this approach. As "pupils at school" they say they are motivated to examine such systems of problems for several reasons:

1. They are excited that through the “construction of an auxiliary circle” approach they have successfully managed with a serious competitive problem.
2. Even without analysis it can be understood, that the additionally constructed circle brings benefits at least in that it creates prerequisites for the use of a large volume of theoretical base – inscribed angles, peripheral angles, angles with a vertex outside the circle or with a vertex inside of the circle, geometrical locations of points and etc., which enlarges to a high degree the opportunity to find a solution.
3. There cannot only be seen and evaluated the great effectiveness of the additional construction – **an auxiliary circle**, but also to be felt the originality and beauty of the solutions themselves of the problems from the system, on the base of this method.

It is worth mentioning the opinion of the students, that they have not only mastered a particular method that can shorten the way for searching and finding out solutions to a particular type of mathematical problems (since in most of them the need of analysis might be ignored) but they have realized in reflexive plan, from the position of "talented" learners "rediscovering" (under the guidance of the trainer) known methods for solving problems, that they can gradually gain experience, on the base of which later, they possibly could find alone their own, original particular methods, and this may contribute the improvement of the methodology of handling with problems.

As a negative side of the approach for "rediscovering" the method of auxiliary circle described above, the students observe the following:

- there is a risk of losing valuable training time because it is difficult to determine the number and the type of problems of the system by which the knowledge and skills for the method can be inserted in the zone of near development (ZND). It is difficult also to determine how many and what kind of problems are needed to include the same knowledge and skills in the zone of actual development (ZAD);
- the approach could not be universal for all students, since for some of them the summary and formalization may turn out to be too hard, while for others they may be too easy.

Finally, we come to the conclusion, that the pointed above forming potential of the considered approach can be utilized to some extent, and the negative sides can be neutralized (also to a certain extent) if the teacher is able to diagnose properly the achievements of the students and dose skillfully the educational instruments for introducing knowledge and skills respectively in the ZND and the ZAD.

It is appropriate to hold a discussion with students (and advanced pupils) on the characteristics of the most often used methods in teaching practice, for example: method of identity transformations (based on the properties of the algebraic operations with numbers and formulas for "short multiplication"); the method of equivalent transformations (based on the theorems about equivalent equations, inequalities and systems of them); the method of inverse operations; the method of mathematical induction (based on the forth Peano’s axiom); the undetermined coefficients method; the method of derivatives for studying functions; the graphic method; the method for finding extremes based on the Cauchy’s inequality; the method for solving equations based on finding and comparing the extreme values; method of enhancing inequalities; the method of similarity; the method of geometric transformations (symmetry, rotation, translation, homothety); solving a right triangle; solving an arbitrary triangle; method of areas of figures; algebraic method for solving constructive problems; method of projection (parallel, orthogonal, central); trigonometric method; vector method; coordinate method and others.

As a result of examining typical applications of the different methods we establish that the common feature of them is that they are particular-mathematical methods that are mastered in a certain sequence, determined by the order of study and adoption of the relevant school content, and their basic function is to reduce the solution of a certain problem to the consideration of a problem of known type for which the method of solving is known (is studied). In particular, most of these methods lead the learner to the usage of mastered knowledge of any element (definition or theorem-sign or theorem-property) from a specific mathematical field or mastered knowledge and skills for application of algorithm for solving problems of some, already discussed type (according to the mentioned above universal strategy). That's why these methods can be assigned to a more general class, named **"methods, leading to known type of problems"**. The last mentioned methods are subjected to a more common strategy in the methodology of education: "from easy to difficult", "from less complex to more complex", etc. For example, the discussed method of the auxiliary angle can be assigned to "the method of substitutions", but on other hand, it transforms the treated problem to a problem if known type. The method of substitutions is a widely known method and is applicable to almost all branches of mathematics. As many other methods, it corresponds to the generally-didactic methods of education that are built on the principles "from difficult to easy", "from mote complex to less complex", "from unknown to famous", etc.

The presented above examples for systems of educational mathematical problems, that are designed to acquaint students with the method of contraposition and the method of auxiliary angle, correspond to the cognitive (intellectual) reflection, since, to some extent, they are constructed in unison with the genesis process of originating of these methods. But this approach requires relatively more training time and therefore could not be used continuously in the acquaintance of the students with the methods described above. Another commonly applied approach to familiarize the students with these methods is subordinated to the idea of praxeological reflection. The main feature of this approach is that the learners are directly introduced (without "rediscovery") to various methods for solving mathematical problems by demonstrating their wide practical applicability. Thus, they are convinced of the relatively high "power" of these methods for solving significantly wider class of problems receiving in this way the necessary motivation for mastering the methods.

In the methodological literature, the last problem is often focused on. For example, in the book[70] (Slavov, 1969), a number of important for the school practice generally-logical and particular-mathematical methods for solving problems in algebra are presented both in theoretical and practical sence. For each method the author has given methodological notes on its applicability (the types of problems for which it is appropriate to use); on the grade of students for which it is appropriate; on the theoretical base that the students need to learn ahead in order to adopt it; on the specific educational, training or developing functions that can be achieved through systems of problems typical for the adoption of the appropriate method. Of all these methodological aspects, particularly important for developing skills in students to perform praxeological reflection is the practically-applied aspect, i.e. the students must not only rationalize the knowledge in abstract plan, relevant to the certain method and the skills for its implementation, but they also have to understand clearly how they can use this knowledge and skills in performing various mathematical activities, they have to realize the visible benefit, the certain implementation, the broad applicability and importance of the appropriate method. This can be achieved through systems of

[70] Slavov, K. (1969). *Basic methods for solving problems in algebra*. Sofia, Bulgaria: Narodna prosveta. [In Bul.]

meaningful, interesting and various enough types of mathematical problems, during which solving the treated method is a key element.

In connection with the improvement of the methodical preparation of students – future teachers in mathematics, the students are given didactic tasks (writing of reports, essays, term papers) that they have to report on seminars during the courses Methodology of teaching mathematics, Methods and methodology for solving mathematical problems, where essentially discussions are performed ending with evaluation of each student's performance. For example, students were given the didactic task to select mathematical problems and structure them in a system in order to demonstrate to pupils at school the broad applicability of the chosen specific methods for solving problems. In one of the discussions, an idea emerged to build and analyze systems of mathematical problems, through which elements of both approaches to be implemented – "rediscovery" and direct demonstration of the broad applicability of the method by few meaningful, interesting, although relatively difficult problems, which must be solved with a visible assistance by the trainer. The general impression was that the more difficult one problem was, requiring a greater tension and being solved successfully, the emotional experience of success was stronger and it delivered more joy. This increases the challenge of entering the "depths" of mathematics by overcoming difficult and more difficult problems. Further we remark briefly some of the main conclusions drawn by students during the discussions.

Some students used the already constructed and discussed system of problems, related to "method of an auxiliary angle", as a base for the implementation of the new idea about compilation of the two approaches. They pointed out that the system of problems contribute to the formation of reflexive skills of cognitive (intellectual) type, since it helped the learners to construct a plan, a scheme, a model, which they can use later for dealing with problems of new types. At the same time, the system of problems can be improved in such a way, that it can serve as a practical approbation of the constructed method. Thus, the assimilation of knowledge about this method reproduces both the model of its discovery and its practical application. According to V. Vassilev[71] (2006), this is one of the characteristics of the realization of the praxeological reflection.

In order to understand more deeply knowledge and skills related to the application of a particular method, aiming to realize a praxeological reflection, it is appropriate to conduct a complex work on the joint application of that method with other productive methods and to emphasize its key role in their fulfilment.

No less important is this work to be carried out by systems of problems with more abundant content. Especially for the method of an auxiliary angle, the students offered to complete the system above with problems 28 and 31 from the book[72] (Merzlyak, Polonski & Yakir, 1994). These are algebraic problems, which are solved by the method of modeling[73] (Fridman, 1998), which, in turn, is implemented by the method of substitutions, thereby their solving is reduced to solving the following trigonometric equations: $3\,sin\,x + 4\,cos\,x = -5$ and $\frac{3}{2}\,sin\,2\alpha - 2\,cos\,2\alpha = -\frac{5}{2}$, the last ones being solved by the method of an auxiliary angle. Using similar appropriately selected examples, learners are convinced that the investigated method is universal for solving not only relatively complex trigonometric equations which can be reduced to equations of the type $a.\sin x + b.\cos x = c$, where the coefficients a, b and c satisfy the inequality $c^2 \leq a^2 + b^2$, but also for solving inequalities of a certain kind, problems

[71]Vasilev, V. (2006). *Reflection in knowledge, self-knowledge and practice*. Plovdiv, Bulgaria: Makros. (p. 192). [In Bul.]

[72]Merzlyak, A.G., Polonski, V.B. & Yakir, M.S. (1994). *Unexpected step or one hundred and thirteen beautiful problems*. Sofia, Bulgaria: Academic press "Marin Drinov", (p. 10). [In Bul.]

[73]Fridman, L.M. (1998). *Theoretical basis of the methodology of education in mathematics*. Moskva, Russia: Flinta. [In Rus.]

connected with transformation of trigonometric expressions, as well as for solving problems concerning extremums of functions, containing trigonometric expressions of a certain type. From the solutions of the cited above problems one concludes that very often when solving a given problem learners use not only the new method but also other methods that have certain relationships with it. Therefore, the emphasis on the practical applicability of the new method, included into one or another system of methods, increases its importance and also leads to a more complete utilization of the forming potentials of these really interesting, beautiful and at the same time multi-functional problems.

In the final stage of this discussion with the students they summarized some key points, related to the mastering of the solving problems methods that are provided in the curricular for the secondary school. Here we will describe some of their generalizations, which according to us are useful for their future career and serve as a basis for their further development and growth as good teachers in math.

The students are acquainted with various bases for classifying mathematical problems and realize not only the theoretical but also the practical significance of the classification based on whether the components "theoretical basis" and "method" of the outer structure of the problem are known or not[74] (Krupich, 1995). According to this classification the problems can be algorithmic, semi algorithmic (semi heuristic), heuristic and creative. Moreover, each of these types has its place, role and importance in the learning process in mathematics. Immediately after studying the subject on any topic, students are offered "simple" problems (training, algorithmic, standard) that they can solve only with the knowledge on this topic. Over time, students have to memorize the different types of problems, the corresponding methods and algorithms for solving, but in order to be able to apply them they must acquire skills to identify to which types of problems a given one belongs. It should be noted that the latter is of essential importance for the correct choice of an appropriate method for solving. In this connection, there is an interesting metaphor, which expresses the analogy between the activity of the doctor and activity of the learner in solving various types (depending on the topic) of standard problems: first a right "diagnosis" must be put, i.e. on the base of the information in the problem the learner has to determine its type and then, according to it to "appoint an adequate therapy", i.e. to choose an appropriate method, algorithm for solving. Very often both the learner and the doctor has to make "reference", using certain reference books (essentially, reference books in mathematics include systems of signs and systems of properties, divided in thematic sections, the knowledge of which and the possession of skill for application, play a key role in the development of the skills for problem solving). In teaching practice, however, in many cases there is a neglect of learning such skills, which leads to a lack of ability to solve the so called "composite" problems, which, in turn – to low self- confidence, feeling of inferiority, low interest in mathematics and ultimately – to lower results.

Together with the methods and algorithms that are closely associated with a particular thematic, the students need to learn also methods associated with more general sections, rather than with specific topics. These methods serve to reduce the solution of certain problems to solving problems of known type (e.g., solving systems of equations by substitution, by summing, by dividing the appropriate sides of the equations, by factorization, by introducing an auxiliary unknown, by reducing to a homogeneous equation, etc.). The knowledge of these methods expands the experience of students in solving mathematical problems and especially their orientation to the approaches and knowledge they have to use.

[74]Krupich, V. (1995). *Theoretical basis of the education in solving school mathematical problems.* Moskva, Russia. [In Rus.]

It is clear, that in solving a *composite problem* a key point is the finding of a suitable approach to guide the learner in using mathematical knowledge that form the "generalized operator" for solving all problems of the given class. Most often it is performed by the already considered basic generally-logical methods, on the base of which the problem is usually "broken" into problem-components with known methods for solving. In applying them, however, they are "joined" with the knowledge of the specific subject. In the process of solving a certain problem, the learner could not find the way in which the thought is "moved" from the given to the sought in the problem, if only generally-logical methods are applied, i.e. without using a certain specific mathematical knowledge, without using certain particular methods. *Consequently, it is necessary to form skills for joint implementation of generally-logical and relevant particular methods for solving mathematical problems.* The realization of this fact by learners is an essential condition for developing their praxeological reflection.

It should be noted, that in solving one and the same problem a variety of methods can be used, each of which can be realized by different operators. The last ones can be distinguished from one another by the input of knowledge (which is related to the complexity of the solution of the problem), and also by the belonging of the specific knowledge to the ZND or ZAD of learners (which is related to the difficulty of the solution). This in turn determines the complexity or the rationality of the implemented solutions.

The most difficult from a psychological point of view, are the Olympic problems for which the "theoretical basis" and the "method" (as components of the information structure of the problem) are outside the ZND of the learner. This requires that in the preparation for participation in the Olympiads the students be acquainted with theorems, methods and other theoretical units, that are beyond the teaching of mathematics in school, to build long lasting skills for appropriate compilation of theoretical knowledge and skills, and various non-standard methods, even to reach the level of creating a new theory and to discover new non-standard methods. As noted in the monograph of S. Grozdev[75] (2007), this is achieved through purposeful solving a large number of thematically diverse mathematical problems with a high degree of problematicity (from different national, not only Bulgarian, competitions and Olympiads, from Balkan competitions and other international competitions and Olympiads).

It is worth noting, that making learners aware of generally-logical and particular-mathematical methods for solving problems can be carried out effectively by two types of systems of teaching math problems, one of them corresponds to the requirements for consistent forming cognitive-reflexive knowledge and skills, and the other – to praxeological-reflexive skills. The reason for this is the realization, that a great part of the described in this paragraph systems of teaching mathematical problems corresponds to and appear to be a simplified model of the "historical" process of differentiation of individual methods and therefore they can be used successfully to form knowledge and skills at the level of cognitive (intellectual) reflection. "Probating" them in the process of solving a variety of meaningful problems from the relevant systems and the understanding that they are applicable for solving other problems outside the system, contributes to their better assimilation on the base of praxeological reflection as well. The creatively thinking person perceives this activity as a methodology that can serve not only as a didactic instrument for mastering through the "rediscovery" of these methods by learners, but also as a model for construction of other methods that are new or unpopular for the students by now.

From methodological reasons it makes sense to comment on the work connected with giving meaning of the basic structural components of the learning activities in

[75]Grozdev, S. (2007). *For high achievements in mathematics.The Bulgarian Experience* (*Theory and Practice*). Sofia, Bulgaria.

mastering specific methods to solve problems. For example, in acquainting students with a type of problems, connected with new material, which are solved using already known method, it is important to update knowledge about this method and skills for its application by several appropriately selected problems from the material which has already been studied, but if possibly of a different type, with emphasis on its general applicability and comprehensiveness. This assists also the creation of motivation for mastering the method. For example, the widely known method of substitutions is applied with great success in solving rational equations and inequalities, and systems of them (of many different types), irrational equations, inequalities and systems of them, transcendent (exponential, logarithmic, trigonometric) equations, inequalities and systems of them, and some types of geometric problems, etc. Using the method of substitutions in each of these algebraic sections and themes, the students have to recall first some of its previous applications, to point out the type of problems in the solving of which it is applied by drawing a *"list"* of the types of problems and the appropriate substitutions, and then to consider the "new" application and to add it to that list.

Conclusions. One of the most important directions to improve the mathematical abilities of learners is training in problem solving. The mastering of different methods and heuristics for finding out the solutions of problems is of significant importance for the achievment of this goal. The methodological models for mastering generally-logical and some particular-mathematical methods for solving problems from school course in mathematics presented here play a role as methodological foundations for the construction of new didactic systems of mathematical problems designed to acquire the essence and applicability of other methods for solving problems. These contribute essentially the extension of the mathematical abilities of learners, which is especially necessary for the full realization of a modern person in the fast developing technological society.

CHAPTER THREE

PRE-SCHOOL MATHEMATICAL EDUCATION

3.1. Mathematical Development of Preschool Children: from Informing to Understanding

H. Brezhneva

Stating the problem. The development of the ways of ensuring mathematical development of children of the preschool age is impossible without realizing the importance of mathematical knowledge in the modern world, its roles for our state and society in general. Mathematical literacy development is an inalienable element of education, culture, social, personal and professional competence. Therefore, there can be no doubt in the relevance and importance of providing the mathematical development of children, beginning with the preschool age.

In Ukraine the importance of mathematical education of children and young people is recognized at a state level. The «State target social program of upgrading school naturally mathematical education on a period to 2015 year»[1] (Cabinet of Ministers of Ukraine, 2011) and Conception of realization of this social program of upgrading school naturally mathematical educations[2] (Cabinet of Ministers of Ukraine, 2010), where a main task of mathematical education on the modern stage is determined as a *forming of national mathematical competence of pupils* confirm this. At the same time, the Conception mentioned has not yet found its eventual actualization in providing school mathematical education. The reasons of such inhibition, in our opinion, are common in the practice of mathematical preparation of preschoolers and pupils of primary school. Mathematics causes most difficulties for children and it is one of the subjects which pupils like less. First of all, it happens because of the discrepancy between the current methodical system of mathematical preparation and the requirements of the society and the production sector. Also, there is a tendency to minimization the mathematics segment in the secondary school curriculum. The teaching of mathematics for preschoolers and pupils of primary school often has a monotonous character as observed in aids, rhythm, rate and methods of teaching which turn it into a banal regulated process; subjects of frontal character hinder the confirmation of pedagogic of collaboration and obtaining the feed-back from a child; the monotony in the application of mathematical games and their low efficiency is observed; lesson-planning is not in-depth, modern facilities, information technologies and active methods of teaching children are not applied efficiently; the sensual mechanisms of perception and processing of information are not sufficiently supported. The consequences of such transfer of mathematical knowledge are the underdeveloped skills of generalization, systematization, analysis, synthesis, classification, seriating and so forth. Obviously, it is worth orienting a child on a comprehension, understanding, and gaining experience, in fact a human is not born with the developed logical thinking, knowledge about the surrounding reality, about the universe, etc. A human masters natural laws on the basis of understanding the logical laws of thought. And mathematics becomes the tool to

---

[1] *State target social program of upgrading school natural and mathematical education for the period till 2015.* (2011). Retrieved from http://dniokh.gov.ua/wp-content/uploads/2014/12/Prirodnicho-matematichna.pdf (in Ukr.).

[2] *Conception of the target social program of upgrading school natural and mathematical education for the period till 2015. № 1720(2010, 27 August).* Retrieved from http://zakon.golovbukh.ua/regulations/8451/467747/ (in Ukr.).

master these laws. Hence, it is possible to assert that the idea of ascent from philosophy of knowledge (informing) to philosophy of understanding must become a basic one for a new philosophy of preschool mathematical preparation.

The aim of our research is to define theoretical methodological principles of the issue of mathematical development of 3-6 year old children, to project and field-test the methodical construct of mathematical development of the preschool children. The followings tasks of research are selected: 1) to define theoretically the methodological bases of mathematical development on the stage of preschool childhood; to specify the maintenance of the key concepts of «mathematical development», «mathematical maturity», «developing the elementary mathematical overview», «mathematical competence», etc.; 2) to describe the psychological mechanisms of the process of understanding and interpretation of mathematical material by preschoolers; 3) to systematize criteria and diagnostic tools of assessing the levels of mathematical development of children of preschool age; 4) to develop and field-test special, organizational and pedagogical, didactic and technological constituents of the methodical construct of the system of preschoolers' mathematical development; 5) to develop and experimentally check the technology of professional preparation of teachers for the implementation of the tasks of the preschoolers' mathematical development. The leading idea of research is incarnated in a hypothesis and is divided into the chain of hypotheses, it is based on the supposition that high-quality mathematical development of preschoolers in the preschool educational establishment environment will be successful in case of: the introduction in the educational process of the methodical construct of preschoolers' mathematical development in the unity of the organizational and pedagogical, and didactical and technological components of the aim; the creation of the sensory-cognitive space of a child by introducing the functional model of mathematical development which is based on the understanding and interpretation of mathematical content; the training of the teachers to the meeting of the challenges of the preschoolers' mathematical development which may be made possible through the use of special training technologies.

Materials and methods. On the stage of the theoretical comprehension of the stated problem scientific methods of theoretical cognition are enacted: method of analysis and generalization of literary sources, scientific, educational, methodical, instructional and normative documentation. On the stage of the study of the high status of the mathematical development of preschoolers, methods of mathematical statistics are applied in the work of the Preschool Educational Institutions of Ukraine.

Analysis of actual research. On the first stage of research an analysis of psychological and pedagogical literature on the mathematical development of preschoolers was carried out. This analysis allowed defining the level of this development. A great number of scientific and methodical approaches to the mathematical training of children are the evidence for the multidimensionality of research in this sphere, both in Ukraine and abroad. Thus, the researches examined the potential of different methods of intensification and optimization of studies of mathematics in different age periods[3] [4] [5] (Galperin, 1994; Davidov, 1997; Kostyuk & Maklyak, 1989; etc.); the ways of forming the concepts of sets and magnitude in children were studied[6] [7] [8] [9] (Althauz & Dum, 1984; Green & Lakson,

---

[3]Galperin, P. Y. & Georgiev, L. S. (1994). *Developing the initial mathematical concepts.* In Z. A. Mihaylova & R. L. Nepomnyashchaya (Eds.), *Theory and methods of developing mathematical concepts of preschoolers : reading-book in 6 parts.* Part 3, 312 p. M: St. Petersburg. (in Russ.).

[4]Davydov, V. V. & Andronov V. V. (1997). Psychological terms of origin of ideal actions. Mathematical abilities of preschoolers. *Psychological science and education*, 3, 27–41(in Russ.).

[5]Kostyuk, G. S. (1989). *Educational process and psychical development of personality.* In L. M. Prokolienko (Eds.). Kiev: Radyanska shkola. P. 300-307 (in Ukr.).

[6]Althauz, D. & Dum, E. (1984). *Color, form, amount. Experience in the development of cognitive abilities of the preschool children.* (V. V. Yurtaykin Trans. from germ.). Moscow: Prosveshchenie. (in Russ.).

[7]Grin, R. & Lakson, V. (1982). *Introduction to the world of numbers.* Moscow: Pedagogika. 193 p (in Russ.).

1982; Papi, 1984; Fidler, 1981; etc.). The ideas of more simple pre-mathematical preparation of preschoolers are realized in the works of A. Stolyar[10] (1988). The features of mathematical preparation of preschoolers in Ukraine have been explored since the 90s of the XX century. The problem is posed in the dissertation studies in such aspects: teaching the features of the concept of time through the models of time[11] (Funtikova, 1992); the combination of different didactic tools for developing the concepts of mathematics' elements[12] (Gaydarzhiyska, 1996); individually differentiated approach to the development of mathematical concepts[13] [14] (Baglaeva, 1997; Stepanova , 2006); cognitive activity, as a factor of mathematical development of senior preschoolers[15] (Brezhneva, 1997) and cognitive independence, as a means for the development of constructive skills[16] (Demidova, 2007); content, form, and methods of developing the elementary mathematical competence[17] (Zaytseva, 2005); teaching mathematical concepts in the process of cognitive activity (S. Tatarinova[18], 2008), computer technologies as means to teach senior preschoolers to count[19] (Pavlyuk, 2012) and others. Besides the dissertation works there are many methodical papers worth mentioning. In them there were developed the approaches to providing logical mathematical development of children[20] (Krutiy & Pletenitska, 2002), the features of the organization of natural mathematical educations of children[21] (Sazonova, 2010), the pedagogical terms of logical mathematical development of children[22] (Mashovets & Stetsenko, 2009) and others. Despite a quite wide variety of research works on the mathematical training of the preschoolers, only separate aspects of mathematical development of children are examined in the majority of them. Quite a large body of research works under analysis touch upon only the senior preschool age. At the level of the dissertation researches the Ukrainian scientists so far do not set the task of creating the integral system of preschoolers' mathematical development. The authors offer only the methodical solutions to some questions of providing the mathematical preparation of children. The analysis of the scientific approaches to providing the mathematical development of preschoolers allowed defining two traditions. One of them is based on the fact that a human (a child) should be able to take advantage of the ready-to-use

---

[8]Papi, F. & Papi. Zh. (1984). *Children and columns.* Moscow: Pedagogika. (in Russ.).

[9]Fidler, M. (1981). *Mathematics in kindergarten.* Moscow: Prosveshchenie. (in Russ.).

[10]Stolyar, A.A. (Ed.). (1988). *Development of the elementary mathematical notions of preschoolers.* Moscow. (in Russ.).

[11]Funtikova, O. A. (1992). *Use of models in developing the preschoolers (of 5-7 years of age) knowledge about time. Candidate's thesis.* Kiev. (in Russ.).

[12]Gaydarzhiyskaya, L. P. (1996). *Developing the principles of mathematical notions in the children of the senior preschool age. Candidate's thesis.* Kiev. (in Russ.).

[13]Baglaeva, N. I. (1997). *Individually differentiated approach to the development of the mathematical notions of the six year olds.* (Master's thesis). Kiev, Institute of Pedagogy APS of Ukraine. (in Ukr.).

[14]Stepanova, T. M. (2006). *Individualization and differentiation of teaching mathematics to children of the senior preschool age.* Kiev: Vidavnichiy dim «Slovo». (in Ukr.).

[15]Brezhneva, E. G. (1997). *Developing cognitive activity of children of the senior preschool age (based on the material of mathematics.* (Master's thesis). Kiev, Institute of Pedagogy APS of Ukraine. (in Russ.).

[16]Demidova, J. O. (2007). *Developing the fundamental principles of cognitive independence of senior preschoolers in the constructional activity.* (Extended abstract of master's thesis). Kiev, NAPS Institute of Problems of Education of Ukraine. (in Ukr.).

[17]Zaytseva, L. I. (2005). *Developing the elementary mathematical competence of the senior preschool age children.* (Master's thesis). Kiev, NAPS Institute of Problems of Education of Ukraine. (in Ukr.).

[18]Tatarinova, S. O. (2008). *Developing logical-mathematical concepts of senior preschoolers in cognitive activity.* (Master's thesis). Odessa, South Ukrainian National Pedagogical University named after K.D. Ushynsky. (in Ukr.).

[19]Pavlyuk, T. O. (2012). *Teaching the computer-backed counting to the children of the senior preschool age*. (Extended abstract of master's thesis). Kiev, NAPS Institute of Problems of Education of Ukraine. (in Ukr.).

[20]Pletenitska, L. S. & Krutiy, K. L. (2002). *Logically-mathematical development of preschoolers.* Zaporizhzhya: TOV Lips. Ltd. (in Ukr.).

[21]Sazonova, A. V. (2010). *General theoretic bases of natural and mathematical educations of the preschool children.* Kiev: Vidavnichiy dim «Slovo». (in Ukr.).

[22]Mashovets, M. A. & Stetsenko, I. B. (2009). *Why preschhol age kids need mathematics.* Kiev: Shkilniy svit. (in Ukr.).

techniques given by the teacher; the other presupposes that, first of all, a child should be taught to think. Thus, domestic traditions were always based on the second tradition, in other words – the basis of mathematical preparation of children was grounded on the task of their intellectual development. It is found out that the supporting postulates of the organization of teaching mathematics to preschoolers were always based on the psychological mechanism of their development, on the leading role of teacher, on the planned format of teaching, on the implementation of the knowledge transfer purpose of teaching, on mastering skills and techniques by the preschoolers[23] [24] [25] [26] (Kozlova, 2003; Leushina, 1974; Tiheeva & Morozova, 1927; Usova, 1981; etc.).

In the course of the analysis of the current state of the problem of preschoolers' mathematical development the body of research of the period from the 80s of the XX century till nowadays has been systematized[27] [28] [29] [30] [31] [32] [33] [34] [35] (Artemova, 1997; Baryaeva, 2005; Beloshista, 2008; Voronina, 2011; Mihaylova & Nepomnyashchaya, 2000; Rihterman, 1987; Smolentseva, 1993; Taruntaeva, 1980; Shcherbakova, 2011; etc.).

The generalization of the material defined certain positions about the ideas of the implementation of the issues of the preschoolers' mathematical development, namely: 1) mastering mathematical concepts and mathematical aids that should prove productive. For this purpose the connection with the idea of intellectual development accompanied by transformations of intellect is a productive one; 2) mastering mathematical concepts should become the mastering of the subject principles, which includes the contents of mathematical concepts formed historically; 3) solution of the main task of ensuring the mathematical development of preschoolers asks for the development of a child as an active subject of cognitive activity, emancipation of the subjectivity of a child; 4) to be focused on child's mathematical development, the approach should employ not only the direct methods of solving various educational tasks, but also, as an integral process, provide the evolution of a child from memorizing of separate bits of knowledge to the understanding of the phenomena of mathematical reality; 5) the approach to teaching mathematics to children is called integral and directed at the effective solution of the main tasks characterized earlier. These conditions show that at such approach the process of teaching mathematics should become a system forming element, based on the processes of forming basic mathematical concepts as effective and strategic instruments of the mathematical activity of a child.

The selection of the afore-mentioned positions stipulated the subsequent ways of the comprehension of the problem of preschoolers' mathematical development. It is known

---

[23]Kozlova, V. A. (2003). *Developing the elementary mathematical notions in the junior children.* (Doctoral dissertation). Moscow, Moscow State University of Education. (in Russ.).

[24]Leushina, A. M. (1974). *Methods of developing the elementary mathematical ideas in the children of the preschool age.* M.: Prosveshchenie. (in Russ.).

[25]Tiheeva, E. & Morozova, M. (1927). *Counting in the life of little children*. Moscow – Leningrad: Gosizdat. (in Russ.).

[26]Usova, A. P. (1981). *Teaching in kindergarten.* Moscow: Prosveshchenie. (in Russ.).

[27]Artemova, L. V. (1997). *Color, form, size, and number.* K.: Tomiris. (in Ukr.).

[28]Baryaeva, L. B. (2005). *Integrative model of mathematical education of preschoolers with time-lagged psychical development.* (Doctoral dissertation). Moscow, Moscow State University of Education. (in Russ.).

[29]Beloshistaya, A. V. (2008). *Development of mathematical capabilities of preschoolers: theoretical and practical issues*. M.: Publishing house of the Moscow psycholodical-social institute; Voronezh: NPO «MODEK». (in Russ.).

[30]Voronina, L. A. (2011). *Mathematical education in the period of preschool childhood: planning methodology. (Extended abstract of Doctor's thesis).* Yekaterinburg, Urals State Pedagogical University. (in Russ.).

[31]Mihaylova, Z. A. & Nepomnyashchaya, R. L. (2000). *Theory and methods of developing mathematical notions in preschoolers: reading-book:* [in 4-6 p.]. (in Russ.).

[32]Rihterman, T. D. (1987). *Teaching notions of time to the preschool children.* M.: Prosveshchenie. (in Russ.).

[33]Smolentseva, A. A. (1993). *Story-based didactic games in mathematics*. M.: Prosveshchenie. (in Russ.).

[34]Taruntaeva, T. V. (1980). *The development of elementary mathematical concepts in preschoolers.* M.: Prosveshchenie. (in Russ.).

[35]Shcherbakova, K. Y. (2011). *Methods of teaching elements of mathematics to the children of the preschool age.* Kyiv. (in Ukr.).

that mathematics plays a specific role in the development of the intellect, as it lays foundation for the further active development of a cognitive sphere of a child. In its turn, the development of mathematical notions is a powerful means of the intellectual development of a preschool child. Consequently, psychological and pedagogical conceptions of intellectual development of preschool child were considered as a base of mathematical development of children. We analyzed different psychological and pedagogical conceptions of the intellectual development of children[36 37 38 39] (Anderson, 1983; Piazhe, 2001; Holodnaya, 2002; Rubtsov, 1996; etc.) and outlined the ways of providing the preschoolers' mathematical development. The necessity of the analysis of different intellectual conceptions is caused by the understanding that the future of a little personality, their role in the society, the possibilities for self-realization and personal prosperity depend on the maturity of the intellect. In the process of the analysis of a source base the concepts of «intellect» and «intellectual development» were specified. It was found out, that the general concept of intellect determines it as a system of psychological mechanisms which predetermine the possibility to build in an adequate model (picture) of outward things into an individual, to organize their behavior and activity through creating order out of chaos on the basis of bringing the individual necessities into conformity with the objective requirements of reality[40] (Elkonin, 1989). The interpretation of V. Shterna[41] (2009) determines an intellect as the «individual's general ability to adapt their thought to the arising requirements». Consequently, an intellect is the original system or ability which helps a human to adjust to the outer factors. Thus, the intellectual development is both a process, and a level of cognitive activity of a growing human in all its displays: knowledge, cognitive processes, capabilities, etc.; it is the result of the influence of circumstances on the life and environment on the child.

During the research different concepts of intellect were analyzed in detail. The first substantial one was the conception of intellect by Jean Piaget, concerning the preschool age children. His theory of intellectual development covers the period from the infant age to maturity. Jean Piaget focuses on the development of a child's thinking, and, first of all, – on the development of logical thinking. The stages of intellectual development (sensorimotor, preoperational, concrete operational stage, and formal operational stage) can be examined as the stages of psychical development in general. Unlike other classifications of psychical development of a child an intellect stood in the center of Piaget's system. The development of other psychical functions at all stages complies with an intellect and it is determined by it.

The analysis of types of intelligence was made according to M. Holodnaya[42] (2002), who distinguished common, convergent, reproductive, and crystallized intelligences. This classification is based on the pair contrasting of intellect functions. The researcher's idea consists in it that the combination of different types of intelligence according to their functions gives the best result of teaching or creation, generating ideas and so on.

---

[36]Anderson, J. R. (1983). *The architecture of cognition. (Cognitive science series, N 5).* Cambridge (Mass.), London: Harvard Univ. Press.

[37]Piaget, J. (2001). *The Psychology of Intelligence.* (Translator, D. E. Berlyne - Translator. Publisher: Routledge). Place of publication: London.

[38]Holodnaya, M. A. (2002*). Psychology of intellect.* Piter. (in Russ.).

[39]Rubtsov, V. V. (1996). Social-psychological conception of the intellectual development of child A.-N. Perre-Klermon. *Psychological science and education,* 2, 20-26 (in Russ.).

[40]Elkonin, D. B. (1989). *Selected psychological labours.* Retrieved from http://psychlib.ru/mgppu/eit/EIT-001-.HTM#Раздел_I (in Russ.).

[41]Shtern, V. (2009). *Psychology of early childhood and up to six years of age.* In E. Minkov (Ed.), *History of foreign psychology from the end of XIX till begining of XX centuries: Reading-book* (p.p. 80-87). Moscow: Flynta: MPSK (in Russ.).

[42]Holodnaya, M. A. (2002). *Psychology of intellect.* Piter. (in Russ.).

From Z. Kalmikova's definition of intelligence it is clear, that the «kernel of individual intelligence is formed by human abilities of the individual of assimilating new knowledge and its application in typical problem situations»[43] (Kalmikova, 1981).

Further analysis of the intellectual conceptions showed that H. Gardner's[44] (1999) ideas, which characterize a human intelligence as multiple, can be a theoretical basis of mathematical development of preschoolers in the process of forming primary mathematical notions. H. Gardner maintained that a human possesses seven different directions of the development of intelligence, he named them the «types of intelligence» and described them in the following way: 1) *verbal/linguistic intelligence* – as the ability of speech acts, which includes the mechanisms responsible for phonetic (sounds of language), syntactic (grammar), semantic (sense), and pragmatic language components (use of language in different situations); 2) *musical intelligence* – as a capacity for creation, transmission and understanding of senses, related to the sounds, including the mechanisms responsible for the perception of the pitch, rhythm, and timbre (qualitative characteristics of sound); 3) *logical-mathematical intelligence* – as the ability to use and estimate a correspondence between operations or objects, when they actually are not present, in other words – the ability of abstract thinking; 4) *spatial intelligence* – as the ability to perceive visual and spatial information, to modify it and reproduce visual images without an access to the initial stimulus. The latter includes the ability to construct images in three dimensions, and to move and flip these images mentally; 5) *bodily-kinesthetic intelligence* – as the capacity to use all parts of body in tasks solving or in products creation. This one also includes control over rough and delicate physical skills and the ability to manipulate external objects; 6) *intra-personal intelligence* – as the capacity to recognize one's own feelings, intentions, and reasons; 7) *interpersonal intelligence* – as the ability to understand and compare feelings, opinions, and intentions of other people. The theory of multiple intelligences of Howard Gardner was selected to be the scientific base of our research. Here is the definition of intelligence that belongs to him: «intelligence – is a bio-psychological potential to process information that can be activated in a cultural setting to solve problems or create products that are of value in a culture»[45] (Gardner, 1999). This concept of intelligence shows a child's specific adaptive resource which becomes a universal intellectual resource. Thus the fundamental propositions of H. Gardner's theory of multiple intelligences (MIT) play an important role for a teacher-educator: intelligence cannot be measured in laboratory conditions with tests; it is impossible to explain racial, national and religious differences on the basis of tests; the intelligence is multiple; the intelligence is dynamic. Two last positions are the most important ones for a teacher as the intelligence *multiplicity* is exactly a possibility to develop a child's intellect in different ways, directions, promoting simultaneously the general level of intelligence. The intelligence *dynamic* bases on the biological rules of the inheritance range of quality features that are why it is necessary to develop the innate abilities of a child, even if they seem to be at the lowest level of development. On the basis of the analysis of H. Gardner's theory it is possible to make up conceptual conclusions: there is a universal system of the estimation of a general level of intelligence for people from different social classes, different cultures; a child's intelligence, as a developed and formed one, can be developed at least in 7 directions; intellectual capabilities of every person are developed to different degrees in these seven directions; the majority of people can develop any type of intelligence; different types of intelligence can

---

[43] Kalmykova, Z. I. (1981). *Productive thinking as basis of learnability.* Moscow: Pedagogika. (in Russ.).

[44] Gardner, H. (1999). Intelligence Reframed: Multiple Intelligences for the 21st Century, Basic Books, Psychology.

[45] Gardner, H. (1999). Intelligence Reframed: Multiple Intelligences for the 21st Century, Basic Books, Psychology. Pp. 33-34.

cooperate; there are many methods of developing the same type of intelligence. The theory of multiple intelligences clarifies and explains one of the psycho-physiological features of a human and scientifically explains those individual differences which are observed in practical pedagogical activity.

The analysis of the different author's approaches to determine the essence of intelligence and features of its development in little children found out some relation of the intelligence development and the processes of child's understanding or misunderstanding of cognitive material, mastering the surrounding reality objects. According to this, the problem of understanding was analyzed as the basis for the preschoolers' mastering of mathematical material, the optimum variant of the interpretation of the term «understanding» was selected on the basis of analysis and comparison of its different analogues in scientific literature; the analysis of the problem of understanding was realized, as general, inter-scientific; the psychological mechanisms of the process of understanding were defined on the basis of psychological and pedagogical research of different authors; the idea of understanding was grounded as a central link of the teaching mathematics to a preschool child as a main condition of a child's adaptation to the surrounding reality.

An appropriateness of the processes of detailing the understanding is also stipulated by the fact that planning of our technology of mathematical preparation of children will rely on the psychological mechanisms of understanding in mastering mathematical contents. In our opinion, the process of understanding serves as a basis in a child's mastering the knowledge about the mathematical reality. A substantial volume of research on the issues of understanding (over 70 sources) has been worked out. The examples of different variants of the authors' interpretations of the concept of «understanding» were cited[46] [47] [48] [49] [50] [51] [52] [53] [54] [55] [56] [57] (Bila, 2011; Diltey, 2001; Doblaev, 1974; Gadamer, 1988; Znakov, 2009; Mirakova, 2001; Korobov, 2005; Kostyuk, 1950; Riker, 2002; Rukosueva, 2010; Kintch & van Dijk, 1978; Rumelhart, 1977; etc.). Consequently, the successful realization of the ideas declared earlier and practical implementation of the results in teaching and education require an access to the problems of ensuring the mathematical maturity of preschoolers. In this context it is expedient to consider the key concepts of *«mathematical development»* and *«mathematical maturity»*.

---

[46]Bila I. M. (2011). Theoretical analysis of the problems of understanding. *Studies and education of the talented child: theory and practice.* K.: Informatsiyni sistemi, issue 5, 71-82.

[47]Diltey, V. (2001). *Hermeneutics and theory of literature.* In V. Bibihina & N. Plotnikova (Eds.). Collected works in six volumes (Vol. 1-6). M.: Dom intellektualnoy knigi.

[48]Doblaev, L. P. (1974). *Developing the techniques of understanding the text . Text and cognitive activity.* Saratov: Publishing house of Sarat. un-ty. Pp. 50-102.

[49]Gadamer, H. - G. (1988). Truth and method. M.: «Progress», 1988.

[50]Znakov, V. V. (2009). Three traditions of psychological research – three types of understanding. *Questions of psychology*, 4, 14–23 (in Russ.).

[51]Mirakova, T. N. (2001). *Didactic principles of the humanization of school mathematical education* . (Doctoral dissertation). Moscow, Institute of General Secondary Education of Russian Academy of Education. (in Russ.).

[52]Korobov, E. (2005). Understanding as a didactic problem. *Moscow psychological magazine*, 11. Retrieved from http://magazine.mospsy.ru/nomer11/s10.shtml. (in Russ.).

[53]Kostyuk, G. S. (1950).). *On the psychology of understanding. Scientific papers of the SRI of psychology.* Vol. 2, pp. 7-57. Kyiv (in Ukr.).

[54]Riker, P. (2002). *Conflict of interpretations: Essays about hermeneutics.* (I. S. Vdovin, Trans.). M.: «Academia Tsentr».

[55]Rukosueva, D. A. (2010). *Features of the visualization-aided perception of mathematical knowledge. Young people and science of the XXI century: Materials of the XI all-Russian (with international participation) theoretical and practical conference of students, postgraduate students, and young scientists* . (Vol. 1). Krasnoyarsk. Pp. 225-227 (in Russ.).

[56]Kintch W. & van Dijk T. A. (1978). Toward a model of text comprehension and production. *Psychological review*, v. 85. pp. 363-394.

[57]Rumelhart, D. E. (1977). Understanding and summarisation brief stories. *Basic processes in reading: perceptions and comprehension. New Jersey,* pp. 265-303.

According to the determination of A. Stolyar & R. Nepomnyashcha [58] mathematical development is «qualitative changes in a child's cognitive activity which happen as a result of forming the elementary mathematical notions and performing logical operations connected with them». In most research papers a concept of «mathematical development» is understood just like this[59] [60] (Baglaeva, 2002; Levchuk & Ermolchik, 2014; etc.). In the works of the Ukrainian scientists the definition of «mathematical development» finds a specification through the word combination «logical and mathematical development». The origin of this definition is connected with scientific explorations of N. Baglaeva. And at the same time, the researcher interprets logical and mathematical development as «qualitative changes in the cognitive activity of a child, which happen as a result of the development of mathematical abilities and logical operations connected with them»[61] (Baglaeva, 1999). The researcher combines two concepts in a unique concept complex: «logical and mathematical development» and «logical and mathematical competence». And *logical and mathematical* competence is determined as «a child's ability to realize on their own (within the limits of the age period) the classification of geometrical figures, objects, quantities; the seriation, i.e. rating according to size, weight, location in space and time; calculation and measuring of number, distance, length, width, height, volume, mass, and time». In our opinion, according to this definition, the basis of a child's development is their ability to operate mathematical concepts and to make operating actions. These are the basic capacities of a preschool child, but they do not completely provide mathematical maturity of a preschool child in particular. Let us explain our position. The concept of maturity is much wider. A new quality of a child's personality – *mathematical maturity* – must become the predictable result of the application of the system of mathematical development, here the process of mathematical development as a specifically organized and directed system, is directed at forming a child's cognitive experience (mathematical knowledge, abilities, skills, intellectual qualities, and spiritual and moral norms) according to the psycho-physiological characteristics and cognitive necessities of a personality at all stages of formation. It is not easy as the component quality has a two class structure: the *1st class of components* – knowledge, ability, and skills that are formed by mathematical means and necessary in activities, life practice and which promote the proper level of the intellectual development; the *2nd class* – mathematical thinking, fundamental principles of the perception of the world; a power of self-realization; spiritual and moral development; mental qualities (calculating skills, linguistic flexibility, spatial orientation, memory, capabilities of reflection, reasoning, speed of information perception and making decision, etc.). Such interpretation of mathematical maturity is based on our conviction that it is impossible to learn mathematics by heart. Knowledge that is not used in practice stays formal, not active, and vain. And vice versa, the knowledge that is made use of, serves as a basis for activities in different cognitive situations. The essence of mathematics is the logical understanding and beauty of mental activities and not «knowledge». Consequently, «rote learning» of mathematics is impossible. The essence of mathematics is unfolded, first of all, not in the «total amount of knowledge», but in a certain quality and style of thinking. Therefore, the path into the world of mathematics shouldn't lie only in

---

[58]Stolyar, A. A. (Ed.). (1988). *Developing the elementary mathematical notions in preschoolers.* Moscow. p. 7, pp. 104-105 (in Russ.).

[59]Baglaeva, N. (2002). *Logical-mathematical development of preschoolers: ways of optimization*. *Teacher's palette*, 2, 12-14 (in Ukr.).

[60]Levchuk, Z. K. & Ermolchik, I. V. (2014). *Theory and methods of teaching elementary mathematical notions to the children of the preschool age.* Vitebsk: VSU imeni P. M. Masherova. (in Russ.).

[61]Baglaeva, N. (1999). *Current approaches to the logical-mathematical development of preschool children*. *Pre-school education . Doshkilne vihovannya – Pre-school education,* 7, 3-4 (in Ukr.).

mastering mathematical knowledge. Teaching children does not require knowledge; it needs understanding of this knowledge. Hence, the philosophy of understanding consists in laying the foundation for children's worldview. The core of teaching mathematics to children should not consist only in rote learning or multiple training, but in the understanding or misunderstanding of mathematical notions and concepts by children. In this context «understanding» becomes a key word. It is exactly that thing which defines the further way of problem comprehension.

Thus, the understanding of this problem becomes the next stage of theoretical comprehension. The general analysis of understanding mechanisms is grounded on some deductions. The problem of understanding is multidimensional and is studied by many sciences: philosophy, philology, physics, pedagogy, psychology, etc. Numerous interpretations of the concept of understanding make the process of comprehension of its essence more complicated. The researchers explain it in different ways; it depends on the context of research. Philosophers examine understanding «as the actualization of the ties between the objects of the real world in their generalized and mediated reflection». Philologists determine understanding as «an ability to give verbal equivalent, to say something in one's own words, to understand the reasoning correctly, and to report the present knowledge about situations and objects» or as «a simplification of idea, translating it into another language», as «the reflection of the text and its understanding in a new context». Psychologists equate understanding with the process of cognition: *understanding* – is «a process of cognition of new and unknown through the already known»[62] (Kostyuk, 1989). Examining the existent theoretical conceptions and experimental directions, we, at the same time, tried to define on our own the place of the concept of understanding in the field of the problem of understanding. And we came to the conclusion, that within the limits of V. Znakov's[63] (2009) conception which better suits our picture of understanding and its mechanisms, there exists *cognitive approach*. Therefore, our model of mathematical development of children is grounded within cognitive approach. We also used the ideas of such psychologists as M. Bershadskiy[64] (2004), V. Doblaev[65] (1974), G. Kostyuk[66] (1989), M. Holodnaya[67] (2002), who also examined *understanding as cognitive procedure, the essence of which is based on the activation of knowledge, and the process of understanding the meaning of what is becoming known*. On the basis of analysis and systematization of the key definitions of the term "*understanding*" we offer our variant of the definition concerning the children of the preschool age: *understanding as the component of thought shows the procedure of building new knowledge based on the earlier experience of a child and is realized in certain forms of understanding: understanding – identification; understanding – hypothesis; understanding – unification.*

Through the analysis of psychological papers it was found out that understanding is a component of thinking, one of the developmental processes. Understanding does not differ too much from an independent psychical mental process; it provides the establishment of the relations between the new exposed connections and connections, presented in the subject. Consequently, for understanding the new material, a subject

---

[62]Kostyuk, G. S. (1950). *On the psychology of understanding*. *Scientific messages of NSI of psychology*. Vol. 2, pp. 7-57. Kyiv (in Ukr.).

[63]Znakov, V. V. (2009). Three traditions of psychological research – three types of understanding. *Questions of psychology*, 4, 14–23 (in Russ.).

[64]Bershadskiy, M.E. (2004). *Understanding as a pedagogical category.* Moscow: Pedagogicheskiy poisk. (in Russ.).

[65]Doblaev, L. P. (1974). *Developing the techniques of understanding the text . Text and cognitive activity.* Saratov: Publishing house of Sarat. un-ty (in Russ.).

[66] Kostyuk, G. S. (1989). *Educational process and psychical development of personality.* In L. M. Prokolienko (Ed.). Kiev: Radyanska shkola. (in Ukr.).

[67]Holodnaya, M. A. (2002). *Psychology of intellect.* Piter. (in Russ.).

(child) should always solve certain intellectual problems, because the formation of new understanding takes place in the process of intellectual activity and it is also its result. If some already known phenomenon or event should be comprehended, the understanding is carried out without full participation of the thought – it is *understanding-reference*. The same forms of understanding come out in those types of intellectual activity in which understanding is a basic psychological boost, where it plays a subsidiary role, and is found as the activity component.

The object of our research is mathematical development of a preschool child, which consists of the achievement of a new-quality level of mathematical knowledge and mastering logical operations. In the process of mastering contents of the mathematical science by a child we find the *understanding* as a key element from that point of view that mathematics is such a field of knowledge which cannot be mastered without perception and understanding of some basic concepts, actions, and connections between elements. In mathematics cramming is impossible, it will always appear ineffective. We can find the confirmation of this idea in educational research works. Thus, M. Bershadskiy[68] (2004) mentioned that «*knowledge and action without understanding can be formed by rote learning and mere imitating...*». Pedagogical experience and practice prove that those teachers, who first of all care about children's understanding of the material studied, achieve success.

We believe that understanding of the cognitive side of the mathematical support of the mechanisms of understanding will serve the basis for the realising of intercommunications and connections between the objects of reality. Examining understanding as a universal description of any intellectual activity (mathematics is always related with intellect work, comprehension, active cognition, etc.) we regard it as the essential condition of mastering mathematical contents and concepts by preschoolers. To understand means to meet some intellectual challenges simultaneously. Thus, psychological mechanisms of understanding mathematical material bring about the necessity of solving intellectual tasks. Such context of understanding becomes the basis in mastering mathematical material by the preschoolers. Understanding arises in the process of cognition of the objects and phenomena of reality. This process of cognition is characterized by the effectiveness of comprehension. Consequently, the connection of understanding with thought and cognition is obvious. Mastering mathematics is impossible without comprehension and active cognition. Comprehension occurs through new knowledge, and therefore, a new level of a child's understanding and recognising mathematical material in images, objects, phenomena, and the items of the surrounding reality. Cognition and thinking, thinking and understanding will be interpreted by us as interconnected processes. Forms and procedure of understanding selected by psychologists (*understanding-recognition, understanding-hypothesis, and understanding-unification)* allow viewing understanding as such that leans against three phases of a child's understanding of the cognitive situation: stage of recognition; stage of hypothetical predictions; stage of the unification of the elements into comprehensible units.

As it has been stated, the object of our research is mathematical development of a preschooler, which is realized through a child's mastering mathematical material through different objects and phenomena of the surrounding reality. The specification of such mathematical development consists not in rote learning of mathematical terms, concepts, numbers, signs, but in conscious operating, activities and cognition of the objects of reality which is full of hidden mathematical contents. The process of picking out the hidden mathematical contents from the objects of reality is based on the mechanisms of understanding. Consequently, the procedure of comprehending

---

[68]Bershadskiy, M.E. (2004). *Understanding as a pedagogical category.* Moscow: Pedagogicheskiy poisk. (in Russ.).

(understanding) becomes that motivating force which will move the mathematical development of a child forward on a new quality level. It stipulates the particularity of the application of the model of preschooler's mathematical development, based on the mechanisms of understanding and takes into account the specifics of mathematics as a field of cognition. The result of understanding is the construction of the situational model of the object which is to be understood. The conception of this object is built both on the basis of the incoming information and on some previous knowledge about the object, and also on the basis of a child's internal information (a notion about an object, aims, reasons, emotions, feelings that are connected with an object of experiencing, etc). Methods of understanding the mathematics content of the object are a generation of hypotheses about its functions, destinations, and properties. Three basic forms of understanding are selected in our model: *understanding-recognition; understanding-hypothesis; understanding-unification*. Each of these forms of understanding arises up on the basis of a child's actions and operations during the solving of some cognitive situation. Three cognitive procedures, three types of intellectual operations and actions became important for a subject of cognition: 1) identification of new elements in the already known material; 2) forecasting, making hypotheses about the past and future of the object or the situation which is to be understood; 3) combining the comprehended elements into a unit. Consequently, we have a two-polar scheme: on one side there is a subject (a child) of understanding, on the other side are the objects of understanding (mathematical phenomena, object, situation, and image). Below (Fig. 1) there is a two-polar scheme of the process of a child's understanding of mathematics content is presented, where «K» stands for a kid, as a subject of understanding; MC is for mathematical CONTENT, as an object of understanding, and between them there is an interdependent connection, mediated by judgments, actions, emotions, feelings, and things like that.

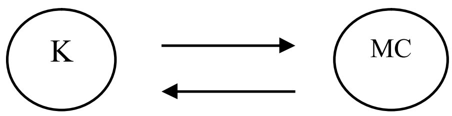


Figure1. Two-polar scheme of the process of understanding of mathematics content

Based on the importance of concepts, knowledge about the object, its functions and properties, a child builds up subsequent logical schemes which provide prompt analysis of the object of cognition (geometrical forms, cognitive situations, etc.). Strategically, the process of making meaningful connections between the objects of cognition passes on to the process of combining the elements of the phenomenon being comprehended into a unit. The mechanism of understanding is organized by the control system, which consists of the object of understanding, the educator and the educational environment organized by them. Thus, there is an object of understanding (K stands for a kid) on the one side of the scheme of understanding, and on the other one there is a control system in general, which determines the essence of mathematical content (MC stands for mathematical content), from the point of the aims and the activities tasks within a subject of understanding, i.e. a child.

We want to comment on it how a mechanism of understanding works. In our opinion, using various games, everyday educational situations (EES), and lessons in teaching mathematics to the preschoolers are a must and this practice must get a new perspective: through the mechanisms of a child's understanding of mathematical content. We will explain it. If we consider the process of teaching children to mathematics as a process of receiving and processing new information (material), suitable for the application in practical life, this information must be clear to a child and included in the system of knowledge that they already have. Consequently, to become clear, educational materials passes a few stages, or rather, areas of processing: 1) perceptibly emotional area, 2) area of memory and 3) imagination. We will explain this mechanism stage-by-stage (Fig. 2).

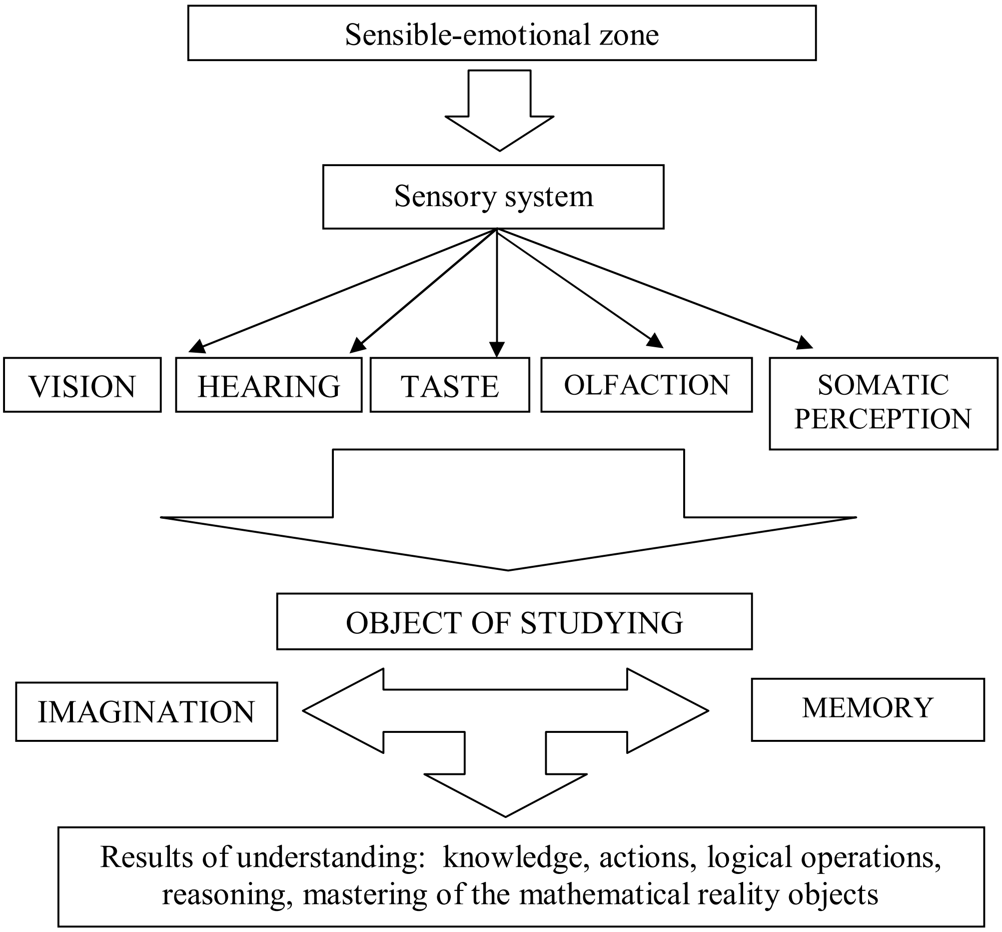


Figure 2. Mechanisms of child's understanding of mathematical content

The first stage – we introduce to the child an image of an object which is studied (it can be an unexpected situation, a mathematical model, an image, etc.). *The second stage* – the emotional sphere of a child is activated through perception which works through the sensory system (vision, hearing, taste, olfaction, and somatic perception). The sensory system is concentrated on the object of study; a child gets complete information about an object through different channels. The multichannel provides getting the first impressions from the object of cognition. *The third stage* – the information received through the sensory system gets into the area of imagination and memory. The result of this is the formation of new knowledge, the implementation of actions, and the application of logical operations. Consequently, the mechanism of understanding of mathematical object is activated. Thus, the mechanism of a child's understanding of mathematical maintenance consists of such sequences: the information comes into a child's brain, at first – into the emotionally perceptible area, then something sparkles in the area of memory, then it is allies with the area of imagination, and at last a certain perceptible image arises. As among preschoolers visual-spatial and visual-active thinking predominate, an educator should apply combination of signals (verbal, perceptible, and tactile) for a deeper understanding and mastering mathematical content by a child, finding it in the objects. Consequently, it is necessary to involve different channels of sensory system; such multichannel influences help a child to understand mathematical content. It is attainable through a game. Thus, games should be rich and various in content and methods of arrangement. In our opinion, naturalness and unexpectedness make children cooperate while playing: through activity, text and interesting tasks. A preschool child does not consciously notice it, but it creates a learning environment for them and allows them enjoy the process itself and the success in the playing activities. That is why it is necessary to avoid instructions like: «Children, today we are going to play an interesting game». It is the most widespread appeal to children with the offer to play. For a child a certain rubber stamp has been formed: «... we shall be taught something again». Meanwhile, the hidden mathematics content is acquired through different variants of its presentation to a child: through a text, independent creative activities, and games. Then, the continuation of the subject serves as a strong foundation of fixing, mastering of knowledge, abilities, skills, logical operations etc. Indirect teaching is the main sense of modern application of playing methods, a game-based form of cooperation. Grounded in sensible-emotional perception, cooperation in pairs or groups and collective games are the formats of children's understanding of mathematical content.

Presentation of basic material. All previously stated is the grounds for creating the technology of children's mathematical development on the basis of their understanding of mathematical content. Such project is offered in this publication. Below there is a scientific rationale of the technology of preschoolers' mathematical development on the basis of understanding and interpretation of mathematical content. Implementing the thesis, declared at the beginning of the article: *«From philosophy of informing to philosophy of understanding»,* let us define the main reference point of technology:

• implementation of the mechanism of the sustained mathematical development focused on the achievement of a new quality of *mathematical development of a preschool child*;

• creation of the educational environment of PEE (preschool education establishments), as the basis of the process of comprehending the mathematical material through the organization of a child's penetration into the essence of relations between the objects of external reality; their active mastering by preschoolers on the different levels of understanding and interpretation using the instruments of cognition and logical reflection of the world «discovered» by children;

• introduction of a unique bio-adequate methodical manual for higher educational establishments and preschool establishments of Ukraine.

Conceptual positions: Integrated didactic module (IDM) is the central structural component of technology. Integrated didactic module technology is such a technology of mastering mathematical material which will enhance knowledge and a child's concepts of mathematical reality through activating their mechanisms of intellectual elaboration of information based on different structures of a child's brain: *brain-action, brain of emotions, and a thinking brain*. All work, organized in the context of IDM, implements the principles of a personality-oriented approach. IDM is a system of mathematical concepts, united on the basis of their semantic connections which create an integral frame. Concepts, relations, and operations gather in pairs, and each of them is studied as one integrated module.

As educational-cognitive construct is united by the general idea of IDM, it takes its stand on five *principles*. We will expose some of them:

1. *Principle of providing cognitive motivation.*

2. *Principle of reversibility – duality – symmetry of mathematical concepts*. This principle of IDM, as an index of mathematical development, it provides the degree of

the intensification of understanding of the essence of the objects and relations between them. The fundamental statement here is that all mathematical concepts are characterized by contrast-duality, symmetry[69] (Erdniev, 2004): «black – white»; «big – small», «night – day»; «heavy – light». The same thing applies to mental actions: «unite-separate»; «add-subtract» «increase-decrease». For every intellectual action there is an appropriate symmetric action which allows going back to an initial point. The reversibility of the idea is a criterion of stable equilibrium in understanding mathematical and logical concepts and connections between them. The essence of it is that a mental action starts from the result of the first action. A child makes a symmetric intellectual action and then this symmetric operation results in the initial state of an object, without changing it. In other words, mastering reverse operations provides an overcoming of initial egocentrism. Then, in the process of conceptual thinking, when there are no restrictions because of the free transfer of the beginning of coordinates (decentralization), the intellectual environment grows that allows to build the system of relations and classes, independent and decentralized according to a child's own «Self». On the pre-conceptual level direct and reverse operations still do not combine in completely reverse compositions that is why the presence of invariance of relations has limits which stipulate the defects of understanding. A child's insensitivity to contradictions serves as basis for it. To form in a child real scientific knowledge and not a simple aggregate of empiric knowledge, it is not enough to provide their physical practicing with mathematical material with the following memorization of the received results. The necessary experience, especially logical and mathematical, is directed at the operations with real objects.

3. *Principle of basing on psychological mechanisms of processes of understanding, as a component of thinking.* Fundamental regularities of thinking are used as an activity. Any intellectual activity or understanding, on a physiology side, is an analytical-syntactical activity of a human brain, where analysis is a selection of the substantial and synthesis is the actualization of connections, formed in the past experience or newly-acquired, which combine indissolubly with each other and ensure the success of understanding. All intellectual processes including understanding are materialized verbally and through activities. The criterion of understanding is a combination of these two factors: verbal explanation of an action and actual implementation of this action (application in practice).

In the process of understanding it is important to ensure the connection between *word and visual images.* A realization algorithm of teaching is built through *three cognitive procedures* (all codes which carry mathematical sense are used – word, picture, physical image, chart, model, practical actions, etc.): *first stage* – presentation of the initial task through an unexpected situation, a story or a text format (*identification* of familiar concepts in new material); *second stage* – producing visual images (*prediction,* making hypotheses about the past and the future of an object); *third stage* – performing practical actions with cognitive material (*combining* familiar elements in a unit).

4. *Principle of taking into account the complicated character of mathematical knowledge, systemic knowledge acquisition.*

5. *Principle of ensuring the unity of understanding procedures through three forms: identification of familiar elements in new material; prediction, making a hypotheses; unification of the understood elements in one unit.*

The central element of the algorithmic procedure of mastering mathematical content is «mathematical exercise», as a variant of combination of child and teacher's activity,

---

[69] Erdniev, P. M. & Mikerova G. G. (2004). *Principles of evidentness, systematic character and sequence in technology of large-sized didactic units at school.* In E. Malinochka (Ed.). Krasnodar, Kubanskiy State University. P. 12-24 (in Russ.).

as an elementary integrity of a bilateral cooperation process «teacher – child». We think that the triple structure of mathematical exercise, the elements of which are examined at one lesson is: 1) the initial task; 2) its reversibility; 3) generalization.

A task, based on multichannel connections, must become the basic form of mathematical exercise.

The following rule serves as the main line of a lesson, built in the IDM format: not repeating but transformation of the task performed at the lesson. In other words, cognition and understanding of the object of learning consist in their development, change and reversibility. And on this basis, an intellectual position of child, which determines their attitude toward reality in general and mathematical content in particular, is being gradually formed.

Conclusions and prospects for further research. On the way of a child's mathematical development they are to study, master and understand everything only on their own, they by themselves are to get the essence of categories and concepts, relations and dependences. Hence, the problems of elaborating deep-laid strategies of providing preschoolers' mathematical development become very important. Mastering logical-mathematical operations and methods of cognitive activity is impossible without developing flexible, coordinated, algorithmic actions, without including mechanisms of understanding and interpretation of mathematical content. All of it must be grounded on various active, emotionally complete activities of a child as a subject of teaching. A child's personality is much more substantial and multidimensional than our idea about it. Therefore, mathematical development of preschoolers should be examined as the enlargement of the development opportunities of a human in general. Such personality oriented model of mathematical development helps a child become socially-active, independent, and a highly intellectual person in future.

We have analyzed the essence and the meaning of the concept «mathematical development» of preschoolers and projected a technological model on the basis of IDM. Further scientific research presupposes experimental introduction of the model of preschoolers' mathematical development in the practice of preschool educational establishments of Ukraine.

CHAPTER FOUR

MATHEMATICAL TRAINING AT SCHOOL

4.1. Developing Cognitive Interest of Secondary School Students in Extracurricular Activity in Mathematics

I. Akulenko & I. Vasylenko

The process of reforming the national education system associated with Ukraine's integration into the European Education Space objectively actualizes the renewal of the goal and content of school mathematical education and consolidates the key strategy focusing on each individual as the highest social value of the educational process. It is declared in the major state documents on education[1] [2] (the Law of Ukraine "On General Secondary Education", 2010; "National Strategy for the Development of Education in Ukraine for 2012-2021"; 2013).

Cognitive interest is a mighty motivating force that guides students in active learning and cognitive activities, making them a fascinating and full of emotions process. The problem of cognitive interest development is considered in the works of modern researchers[3] [4] [5] [6] [7] [8] [9] (J. Piaget, 2001; S. Radko, 1978; S. Rubinstein, 1999; G. Shchukina, 1988; R. Cettel, 1971; A. Fabre, 1978; B. Lopez, G. Frederick, R. Lent, W. Brown, & D. Steven, 1997; etc.). The analysis of psychological and pedagogical studies shows complex and multidimensional nature of the concept "*cognitive interest*". The term is interpreted as a complex individual formation concerning various mental processes[10] [11] (L. Lokhvytska, 2000; S. Radko, 1978); an integral personal characteristic encouraging an individual to constant search of a way to transform a reality by means of activity[12] (G. Shchukina, 1988); an effective motive for learning and education activity[13] [14] (B. Bondarevskyi, 1969; B. Druz, 1978); a student's need to acquire knowledge[15] [16] (L. Bozhovych, 1966; N. Morozova, 1969); a stable trait of a student's

---

[1] *The Law of Ukraine "On General Secondary Education"*. (In Ukr.). http://zakon2.rada.gov.ua/laws/show/651-14.

[2] *National Strategy for the Development of Education in Ukraine for the period until 2021, approved by Decree of the President of Ukraine of 25.06.2013 number 344.* (In Ukr.). http://zakon4.rada.gov.ua/laws/show/344/2013

[3] Obuhova, L.F., Burmenskaya, G.V. (Eds.) (2001). *Jan Piaget: Theory. Experiments. Discussions*. Moscow: Gardariki. (In Rus.).

[4] Radko, S. A. (1978). *Developing cognitive interest in the Process of Educational Activity*. Minsk: Nauka i Tekhnika. (In Rus.).

[5] Rubinstein, S. L. (1999). *Fundamentals of General Psychology.* In 2 parts. (2nd ed.). SPb.: Pier Com. (In Rus.).

[6] Shchukina, G. I. (1988). *Pedagogical Problems of Developing Students' Cognitive Interests: Scientific Edition.* Moscow: Pedagogika. (In Rus.).

[7] Cettel R. B. (1971). *Abilities: the structure grows and action.* Boston: Noughtoun Mifflim. (In Eng.).

[8] Fabre, A. (1978). *Scientific pedagogy and education.* Parise: Libr. Colin. (In Fr.).

[9] Lopez, B., Frederick, G., Lent, R., Brown, W., & Steven, D. (1997). Role of social-cognitive expectations in high school students' mathematics-related interest and performance. *Journal of Counseling Psychology. Vol. 44 (1), Jan.* 44–52. (In Eng.).

[10] Lokhvytska, L. (2000). *Developing Cognitive Interests of Senior Children in the Learning-through-play Environment.* (Abstract of the PHD Dissertation). M.P. Drahomanov National Pedagogical University, Kyiv, Ukraine. (In Ukr.).

[11] Radko, S. A. (1978). *Developing cognitive interest in the Process of Educational Activity.* Minsk: Nauka i Tekhnika. (In Rus.).

[12] Shchukina, G. I. (1988). *Pedagogical Problems of Developing Students' Cognitive Interests: Scientific Edition.* Moscow: Pedagogika. (In Rus.).

[13] Bondarevskyi, V. B. (1969). *Interest in Knowledge and in Self-Education. Developing Students' Striving to Self-Education.* Volgograd: VGPI. 51-69. (In Rus.).

[14] Druz, B. G. (1978). *Training Cognitive Interests of Younger Students in Education Process*. Kyiv: Radianska Shkola. (In Ukr.).

[15] Bozovic, L. I. (1966). *Age Laws of Forming Child's Personality*. (Abstract of Doctoral Dissertation). Moscow, Russia. (In Rus.).

character[17] (N. Bibik, 1997); an important means of education[18] (T. Sushchenko, 1970); an individual's selective focus on mastering knowledge in a particular subject area [19] (N. Kulchytska, 1999).

The leading Ukrainian Methodists and Mathematicians M. Ihnatenko[20] (1997), A. Kukhar[21], (1984), O. Skafa, E. Vlasenko, & I. Goncharova[22] (2003), Z. Sliepkan[23] (2000), N. Tarasenkova[24] (2002), L. P. Cherkaska, O. I. Matias, & V. A. Marchenko[25] (2014), etc. studied some aspects of the formation and development of cognitive interest in teaching Mathematics. This problem was considered by I. Garifulina[26] (2003) (pedagogical conditions of developing cognitive interest of primary schoolchildren in the interaction with nature), O. Grebennikova[27] (2005) (project activity as a means of cognitive interest development of senior schoolchildren), Z. Druz[28] (1997) (nonstandard tasks as a means of stimulating cognitive interest of primary schoolchildren), N. Eguliemova[29] (2003) (geometrical tasks as a means of developing cognitive interest of high school students), N. Zhytieniova[30] (2009) (the formation of cognitive interest of 7-9th form students in learning the natural science and mathematics with computer support), A. Kukhar[31] (1984) (forming cognitive interest of students in teaching Mathematics in the 4-7th forms), S. Shumyhai[32] (2013) (the development of cognitive interest of high school students to learning Mathematics by means of science history), etc.

Scientists persuasively prove that classes of Mathematics have a powerful potential in terms of students' cognitive interest. At the same time, the described scientific facts and materials of the research survey of the teachers give grounds to believe that nowadays students' interest in mastering Mathematics has a tendency to weakening. We distinguish the following possible reasons: 1) overloading the students with actual

---

[16]Morozova, N.G. (1969). *Forming Cognitive Interest of Abnormal Children (in comparison with the norm).* Moscow: Prosveshchenie. (In Rus.).

[17]Bibik, N. M. (1997). *Developing Cognitive Interest of Younger Students*. Kyiv: IZMN. (In Ukr.).

[18]Sushchenko, T. I. (1970). *Training Cognitive Interests of Adolescents in Extracurricular Activity.* Kyiv: Radianska Shkola. (In Ukr.).

[19]Kulchytska, G. V. (1999). *Cognitive Interests of Senior Students*. *Practical Psychology and Social Work*, 6, 21-23. (In Ukr.).

[20]Ignatenko, M. J. (1997). *Activation of high school students' learning activities in studying mathematics*. Kyiv: Tyrash. (In Ukr.).

[21]Kuhar, A.V. (1984). *Developing Students' Cognitive Interest in Learning Mathematics in the 4-7th Forms.* (PHD Thesis). Kyiv State Pedagogical University, Kyiv, Ukraine. (In Rus.).

[22]Scafa, E., Vlasenko, E., & Goncharova, I. (2003). *An integrated approach to the development of a creative personality through a system of heuristic mathematics tasks*. Donetsk: Company TEAH. (In Ukr.)

[23]Sliepkan, Z.I. (2000) *Methods of teaching mathematics*. Kyiv: Zodiac ECO. (In Ukr.).

[24]Tarasenkova, N. (2002). *Theoretical & methodical principles of the use of the sign-symbolic means in teaching mathematics of the basic school students.* Cherkasy: Vidlunya Plus. (In Ukr.).

[25]Cherkaska, L.P., Matias, O.I., & Marchenko, V.A. (2014). *The problem of the Development of the pupils' cognitive interest.* Poster session presented at the International Distance Methodical Conference "ITM * plus – 2014", Sumy, Sumy State University, p.p. 43-45. (In Ukr.).

[26]Garifullina, I. V. (2003). *Pedagogical Environment for Developing Cognitive Interests of Senior Schoolchildren in Interaction with Nature* (Abstract of the PHD Dissertation). Surgut State University, Surgut, Russia. (In Rus.).

[27]Grebennikova, O.A. (2005). *Design Activity as a Means of Developing Cognitive Interests of Senior Students*. (Abstract of the PHD Dissertation). Novgorod Yaroslav the Wise State University, Veliky Novgorod, Russia. (In Rus.).

[28]Druz, Z.V. (1997). *Nonstandard Tasks as a Means of Stimulating Cognitive Interests of Senior Students*. (Abstract of the PHD Dissertation) Taras Shevchenko National University, Kyiv, Ukraine. (In Ukr.).

[29]Egulemova, N.N. (2003). *Modification of Geometric Problems as a Means of Developing Cognitive Interest of Basic School Students.* (Abstract of the PHD Dissertation) Ministry of Education of the Russian Federation, Orel, Russia. (In Rus.).

[30]Zhytieniova, N.V. (2009). *Developing Cognitive Interest of the 7-9th Form Students in Teaching Natural Science and Mathematics with Computer Support.* (Abstract of the PHD Dissertation) G.S.Skovoroda National Pedagogical University, Kharkiv, Ukraine. (In Ukr.).

[31]Kuhar, A.V. (1984). *Developing Students' Cognitive Interest in Learning Mathematics in the 4-7th Forms.* (Abstract of the PHD Dissertation) Kyiv State Pedagogical University, Kyiv, Ukraine. (In Ukr.).

[32]Shumyhai, S.M. (2013). *Developing Cognitive Interest of High School Students in Teaching Mathematics through the History of Science.*. (Abstract of the PHD Dissertation) M.P. Dragomanov National Pedagogical University, Kyiv, Ukraine. (In Ukr.).

information from other disciplines (50% of respondents); 2) reducing the number of Mathematics lessons (45%); 3) insufficient implementation level of the applied orientation of teaching Mathematics in high school (65%); 4) decreasing the prestige of professional activities associated with the use of mathematical knowledge (39%); 5) increasing the share of the humanities among academic disciplines in high school (48%). The weakening of interest affects the quality of mathematical preparation of high school students. In this regard, there is an urgent need to develop modern methods of teaching Mathematics, which is aimed at forming cognitive interest of high school students in Mathematics science and in the process of mastering it in both classroom and extracurricular activity.

The problem of extracurricular activity is considered in the works of I. Honcharova[33] (2006) (methods of forming heuristic skills of high school students at the elective courses in Mathematics), O. Hubachov[34] (2009) (didactic conditions for the individualization of pre-profile training of high school students), R. Yesayan[35] (2005) (forming schoolchildren's interest to self-education in extracurricular activity), V. Zahryvyi[36] (1999) (forming senior students' cognitive interest to economic knowledge in extracurricular activity), L. Lutchenko[37] (2003) (the organization of independent educational cognitive activity of the 7-8th form students while studying Mathematics), M. Melnyk[38] (2005) (forming professional knowledge of vocational school students in extracurricular activity in the Science and Mathematics disciplines). Researchers have made a significant contribution to the theoretical understanding of the problem of improving extracurricular activity in Mathematics; however, practice shows that extracurricular activity in Mathematics often has a number of shortcomings: it is not systemic; its content duplicates the course material mastered at the lessons, or it has no link to the course material; extracurricular activity is of a compulsory nature; organization forms and methods duplicate those ones applied at the lessons; a lack of modern informative and communicative means of education; weak support of social and mathematical experience of students.

Thus, the necessity to form high school students' cognitive interest as an activator of learning and cognitive mathematical activity of schoolchildren, on the one hand, and insufficient development level of the means aimed at forming their cognitive activity in extracurricular activity in Mathematics, on the other hand, give rise to a contradiction to be solved and motivate the research on the problem of Forming Cognitive Interest of High School Students in Extracurricular Activity in Mathematics.

The research is based on the interpretation of the concept "*cognitive interest*" in teaching Mathematics as a selective orientation of an individual that is directed towards a subject (Mathematics), its components and the process of mastering knowledge in Mathematics. Cognitive activity is a necessary condition of cognitive interest development. The interest is formed in the process of conscious activity through awareness and generalization; independence of performance serves as one of the indicators of cognitive interest development. At the level of subjective experience,

[33] Honcharova, I.V. (2006). *Methods of Developing Student's Personality in Mathematics Heuristic Electives. Bulletin of the Cherkasy University. Series: Pedagogical Sciences, 93,* 30-35. (In Ukr.)..

[34] Hubachov, A.I. (2009). *Didactic Conditions of Individualization of Pre-Profile Training of High School Students.* (PHD Thesis) Pedagogy Institute of Ukraine APS, Kyiv, Ukraine. (In Ukr.).

[35] Yesayan, R.S. (2005). *Developing Adolescents' Interest in Self-Education in Extracurricular Activity.* (PHD Thesis) Institute of Preschool Education, Moscow, Russia. (In Rus.).

[36] Zahryvyi, V.I. (1999). *Developing Senior Students' Cognitive Interest in Economic Knowledge in Extracurricular Activities.* (Abstract of the PHD Dissertation) Pedagogy Institute of Ukraine APS, Kyiv, Ukraine. (In Ukr.).

[37] Lutchenko, L.I. (2003). *Planning Unaided Learning Activities of the 7-9th Form Students while Teaching Mathematics.* (PHD Thesis) M.P. Dragomanov National Pedagogical University, Kyiv, Ukraine. (In Ukr.).

[38] Melnyk, M.V. (2005). *Developing Professional Knowledge of Vocational School Students in Extracurricular Activity in Science and Mathematical Disciplines.* (PHD Thesis). Vinnytsia M. Kotsiubynskyi State Pedagogical University, Vinnytsia, Ukraine. (In Ukr.).

interest can be fixed as an emotional coloration acquired in the process of cognition. Cognitive interest is a special activator in the system of various learning motives as it is not only formed in the activity simultaneously with the formation of a personality under the influence of internal and external factors, but, in a sense, it forms an activity being its internal resource and activator.

The content of interest can be expanded and deepened in the educational cognitive activity. Emotional and volitional components of interest interact specifically as an intellectual desire and effort associated with overcoming intellectual difficulties. The conclusion is that cognitive interest takes the following stages of development: on the first stage it is characterized by the emotionally selective focus and instability; on the second stage it becomes more concentrated and resistant form encouraging a student to cognition; on the third stage it is characterized by resistance, cognitive activity and independence of the students, and on the fourth stage – by learning and developing complicated theoretical science issues and their application at practice.

On the basis of the above mentioned statements, we distinguish three levels of cognitive interest development: low, middle and high. We choose the following criteria for their determination: individual activity, awareness of the activity, independence in performance, emotional connection of an individual with a subject of the activity.

The *low level* of cognitive interest has some indicators: weak activity is accompanied by the inaction in problem situations; the awareness level of activity way is low; students realize some notions and facts without understanding their logical relationships; they perform some requirements to the organizations of the unaided work only partly and fragmentary; they are unable to consciously use the instruction to the task; they do not attempt to check the performance; they need a teacher's substantial accompanied assistance, particularly, of corrective nature; there is no positive and emotional marking.

The *middle level* of cognitive interest development is characterized with the following features: students' activity is stimulated and supported by a teacher in educational cognitive activity; the students do not clearly determine the goal of the activity, understand some facts, notions and the relationships; there are some difficulties in the proceedings of performing the activities; there is no transferring of activity patterns to new situations; there is no focus on the unaided activities; reproductive activities dominate; search activities are rarely predominant; the teacher directly manages such activities; some activity types generate students' negative emotions directed at the educational and cognitive activity.

High level of the cognitive interest development is characterized by: a high degree of students' activity (as researchers), search and creative nature of any kind of educational and cognitive activity; the highest level of independence; unaided activities of students are not limited only to the performance of tasks, cover almost the entire educational process and, occur in different combinations of reproductive and research work; students are able to find the means of solving the problem independently, to acquire knowledge necessary for the performance of practical tasks; the internal organizer of students' behavior are emotions, i.e. if students are always in a positive emotional vein, it is accompanied by a high level of mastering the instructional material through the entire educational process.

Extracurricular activity is an organic part of the educational process at school and has a strong potential in meeting the challenges of developing students' cognitive interest to Mathematics. In the research, the concept "extracurricular activity" is defined as a system of special lesson forms in the after-class time based on the principle of voluntary participation and aimed at raising the level of students' mathematical development through deepening and expanding the basic content of the program.

Pedagogical preconditions of developing the cognitive interest in extracurricular activity in Mathematics presuppose: personal development in extracurricular activity in Mathematics; dynamic character, orientation at an individual, validity and control of the process of forming cognitive interest; the combination of sensual, logic aspects and practice in extracurricular activity in Mathematics; the unity of external (pedagogical) and internal (cognitive) activity; the predetermination of developing cognitive interest with age and psychological characteristics being characteristic of adolescent students; the progressive character of cognitive interest and educational and cognitive process in extracurricular activity in Mathematics; the integration of the processes in cognitive, affective, motor-activity spheres of an individual's performance in extracurricular activity in Mathematics; the formation and development of students' general educational, intellectual, and organizational skills; attracting students to productive, personally significant and socially important practical mathematical activity.

We distinguish the following main methodical requirements to the objectives, content, process, and results of extracurricular activity in Mathematics aimed at forming students' cognitive interest: focus on forming the key competences, particularly, mathematical competence in extracurricular activity in Mathematics; attracting personally relevant and socially significant topics, deepening or expanding basic education path of teaching Mathematics; coordinating the content of educational material with social experience of students; supplementing it with information on the history and culture of native land; reflection and positive emotional and valuable marking of mathematical educational and cognitive activity of students; observing the principles of individualization and differentiation in extracurricular activity in Mathematics.

The methodical requirements to the organization of extracurricular activity in Mathematics aimed at forming cognitive interest stipulate the features of constructing every component of methodical system of teaching Mathematics: objectives, content, organizational forms, methods, and means of education.

The main *tasks* of extracurricular activity in Mathematics traditionally involve: 1) awakening and developing students' cognitive interest in Mathematics; 2) the deepest understanding of the important ideas of Mathematics and their ideological functions; 3) assistance in learning the main methods of Mathematics; 4) developing students' mathematical abilities, logical thinking, spatial representations and imagination, algorithmic culture, memory, etc.; 5) forming students' skills of research activity; 6) developing positive features of an individual (mental activity, cognitive independence, needs in self-education, ability to adapt to changeable conditions, initiative, creativity, etc.) and skills to work independently and creatively with educational and popular scientific literature in Mathematics.

Modern educational process focuses on competence; therefore, the following new tasks of extracurricular activity in Mathematics become relevant: to integrate students' personal qualities with content and procedural basis of learning (educational competence); to form an individual's ability to live in a society, to consider the interests and needs of different groups, to follow social rules and regulations, to collaborate with various partners (social competence); to develop ideas about mathematics as part of human culture, the culture of an individual and the society as a whole (general cultural competence); to direct students to preserve their physical, social, mental and spiritual health (health preserving competence); to teach students to orientate in the information space, to retrieve and operate information in accordance with the labor market needs through information and communication technologies (information competence); to accumulate students' potential in the political life of Ukraine, protection of their rights and freedoms, the performance of public duties; to foster love to their land, concern in the problems of building and development (civic competences).

The current content of extracurricular activity in Mathematics extends and deepens the curricular material. The updated content of extracurricular activity in Mathematics should be added to the traditional one, which is reflected in the main lines of content, integrating the information related to the history and culture of the students' native parts. In this regard, new experience gained in the process of solving mathematical problems is organically combined with available scientific and mathematical, historical and mathematical, social and mathematical experience of students[39] [40] (I. Akulenko, 2013; I. Yakymanska, 2000). New information concerning mathematical abstractions complement the pre-formed students' knowledge and interact with the historical, literary and cultural knowledge. In addition, the involvement of the local history material into the students' mathematical activity contributes to its positive emotional and evaluative labeling.

Traditional teaching methods (explanatory, illustrative, reproductive, problem statement, part-search or heuristic conversation, and research) in extracurricular activity in Mathematics should be supplemented with the following ones: project method, the method of heuristic guidelines, the method of appropriate tasks, teaching-through-playing methods, etc. Interactive methods, research and experiment being under didactically prudent control of a teacher are of significant potential for forming cognitive interest.

The society strongly affects the range of students' interests; therefore, the forms of extracurricular activity in Mathematics, which are traditionally used at schools to create educational interest, are affected and require updating. One of the innovative organizational forms of extracurricular activity in Mathematics focusing on forming cognitive interest is historical and cultural mathematical quest[41] (I. Vasylenko, 2015). *Quest* is an amateur mobile intelligent competition, during which participants receive tasks in the form of a route map. The sequence of passing the map along with the correct implementation of the proposed tasks enables success and victory in the competition. The quest characteristics are the following: 1) the quest provides not static form of learning but the mobility of the participants (on-site tours, reviews, etc.); 2) the quest requires the routing map (sequences of tasks, the successful implementation of the previous one is the precondition of implementing the next one and, therefore, passing the entire route map).

The contents of the quests can be different. The tasks provide the students' performance of different types of mathematical activity; therefore, the quest is called mathematical. However, the implementation of a purely mathematical activity for the students with low cognitive interest is complicated and inefficient. At the same time, they are often interested in the activities related to the study of universal and local history and culture. The contents of mathematical quests should involve historical and cultural material. Such quests are qualified as historical and cultural mathematical quests.

We have proven the relevance of the quest tasks based on the native parts' history- and culture-related local materials; for example, the content of historical and cultural mathematical quest "Golden Horseshoe of Cherkasy region" includes the developed tasks in accordance with the objects of the same tourist route. Methodical requirements to the content, the stages of preparing and conducting the quest have been theoretically substantiated and experimentally checked in the research.

---

[39]Akulenko, I.A. (2013). *Theoretical and Methodogical Principles of Developing Methodical Competence of Prospective Mathematics Teacher of Specialized School.* (Doctoral Dissertation). Cherkasy Bohdan Khmelnytsky National University, Cherkasy, Ukraine. (In Ukr.).

[40]Yakymanska, I.S. (2000). *Technology of Personality Oriented Teaching in Modern School.* Moscow: Sentiabr. (In Rus.).

[41]Vasylenko, I.O. (2015). *Development of Basic School Students' Cognitive Interest in Terms of Extracurricular Activities in Mathematics.* (PHD Thesis) Bohdan Khmelnytsky National University of Cherkasy, Cherkasy, Ukraine. (In Ukr.).

To conduct the quest successfully, a teacher should find the historical and cultural materials concerning the places of interest represented in the tourist route "The Golden Horseshoe of Cherkasy Region". This tourist route includes the cities of Cherkasy, Uman, Kaniv, Chyhyryn, Korsun-Shevchenkivskyi, and Talne, the villages of Chevchenkove, Budyshche, Moryntsi, Mezhyrich, Moshny, and Lehedzyne. It is known to be that offers an opportunity to get acquainted with the history and culture of the Cherkasy region.

Passing (virtually) the paths of this route, the students visited various sights of the Cherkasy region, got to know historical facts, information about the outstanding personalities who made a significant contribution to the history and culture of the native land. The items of the mathematical quest correspond to the items of this route.

The route map of the quest was formed of the problems (stations), which should be solved successively. The problems have mathematical, historical, and cultural components. Mathematical component should be consistent with the sections of school course of Mathematics (simple and composite numbers, the greatest common divisor, the least common multiple, ratio, proportion, equations, solving problems with equations, percentages, circumference and circle, rectangle and its square, etc.), Algebra course of the basic school (monomials, polynomials, formulas of the reduced multiplication, rational fractions, square roots, formula of roots of quadratic equation, Vieta's formulas, solving problems using rational equations, random event, probability of random event, arithmetic and geometric progressions, etc.), course of Geometry of 7-9^{th} forms (Pythagorean theorem, trigonometric functions of an acute angle of a right triangle, polygons, vectors, etc.). Local lore historical and cultural aspects of problems reflect historical facts, events, and personalities.

Before solving the problems, the students learn the local lore materials concerning certain places of interest. For this purpose, there are visiting tours to the sights of "The Golden Horseshoe of Cherkasy Region". If the tour is impossible, the students are offered a virtual excursion using presentation (Fig. 1, 2).

Figure 1. Title of Presentation

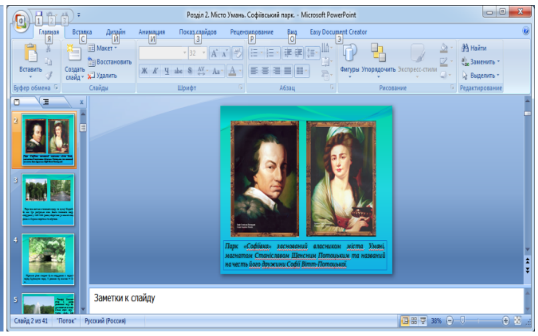


Figure 2. Slide of Presentation

Then the students worked on their own task, isolating the known and unknown historical and mathematical facts in it. After that, they started to solve a number of mathematical problems.

Here is an example. One of the points of the tourist route is the town of Uman, including the famous dendrological Sofiia Park. While visiting the Sofiia Park, the students are offered a route map, according to which every student makes up an individual set of problem tasks that can be solved individually or in a group (independently or with the help predefined).

Route Map (Sofiia Park).

1. Determine what the territory of Sofiia Park is (See Station 1).
2. How much money was spent to build the Park (See Stations 2, 3)?
3. When did the Park move on hold of the Main School of Gardening of Russia (See Station 4, 5)?

4. When was the Park declared as a National Reserve (See Station 6)?

Problem 1 (station 1). Calculation.

What is the area of "Sofiyivka" if during the last 10 years, the area of the former estates of Count Stanislaw Potocki was increased by 6,44 hectares each year (on average), and before that it was 122 hectares?

Answer. The area of Sofiia Park is 180 hectares.

Problem 2 (station 2). Calculation.

The total cost for building the park was 2,000,250 silver roubles (15 million PLN). Count, what was the exchange rate of one rouble for Polish Zloty at that time.

Answer. 1 ruble is 7.5 PLN.

Problem 3 (station 3). Calculation.

The dominant feature of the Park is considered to be a beautiful Low Lake. It is the famous fountain "Snake". If a node of the snake is stretched in length, it will reach the size of 10.65 m, the height of fountain will equal to 25.65 m. What is the height of the fountain jet? What material was the snake cast of?

Answer. The height of the fountain is 15 m; the fountain of "Snake" was cast of bronze.

Problem 4 (station 4). Rounding decimal fractions.

In what year did the Park move on hold of the Main School of Gardening of Russia, if it was in the XIX century, and the third and the fourth digits of the year are known to be a number, which you can get, rounding decimal fraction 59.4006 to the category of whole units.

Answer. 1859.

Problem 5 (station 5). Calculation. Volume and surface of rotation bodies.

The underground river of Acheron was built in the first period of the Park construction, its length below ground is 211 m. The height and width of the underground channel is 3 m. The depth of the water in the channel reaches 1 m.

5.1. Learn how many hatches were fitted throughout the underground channel if they are placed every 53 meters; what functions do they perform?

Answer. 5.1. $211:53=4$ hatches were fitted, through which light reaches the underground the aeration is made possible (natural ventilation).

5.2. How many liters of water does the channel contain if it dives under water (use a formula $V_{цил}=\pi r^2 H$)?

Answer. 5.2. $V_{цил}=\pi r^2 H=\pi\cdot 1{,}5^2\cdot 211=474{,}75\cdot\pi\ (м^3)$

Problem 6 (station 6). Square equations.

In what year was the Park declared as a national reserve, if the first and third digits of this number are serial numbers, and the second and fourth (each of them) digits are the product of the roots of $x^2-11x+9=0$ equation?

Answer. 1929.

Thus, the content of educational material for extracurricular work in mathematics is advisable to develop through its enrichment with the information on the history and culture of the native land. New information concerning mathematical abstractions, compliments the previously formed knowledge of students and is associated with the knowledge of history, literature, and culture and provides emotional and evaluative colouring to the students' mathematical activity. The involvement of such innovative forms of work organization, as a historical and cultural mathematical quest, fosters cognitive interest of secondary school students.

4.2. Conceptual Idea for Arranging Profession-oriented Mathematics Teaching to High School Students in Specialized Schools

I. Lovianova

Over the past three decades, all developed countries have committed and still continue to carry out the reforming of educational systems. Thus the primary aim of the reformation initially raised was to increase the intellectual potential of the nation as well as the development of a creative personality.

The quality of mathematics training of young generation is an indicator of the society's readiness for socio-economic development, the implementation of high technologies, and personality mobility. Mathematical training has been an important component of general education. The task of mathematics in the school system is determined by its role in the intellectual, social and moral development of a personality, in understanding the principles of edifice and use of modern technologies, innovative information technologies, the perception of scientific and technical ideas as well as the formation of a scientific world outlook of the today's school-leavers.

The basic state documents on education in Ukraine: the State National Program "Education. Ukraine XXI Century"[1] (1993); laws of Ukraine "On Education"[2] (2008), "On General Secondary Education"[3] (2010); "Concept of subject-specialized education in high school"[4] (2009); the National Doctrine for the Development of Education[5] (2002); "The National Strategy of Education Development in Ukraine for 2012-2021"[6] (2011) put the emphasis on personal development of a student able to acquire knowledge independently and live in the replete information environment.

The potential of mathematics allows not only forming logical thinking, foster critical thinking and intuition, and affecting intellectual development, but it also helps bring up a person's attitude to mathematics as a part of human culture, the part that plays a special role in social development. This determines the priority of mathematics in developing not only important human traits, but also in the developing of mathematical culture of a school leaver, as a part of their general cultural development, regardless of the future profession chosen. Consequently, the priorities of mathematics education should be as follows: there should be personal orientation in education, integral components of mathematical science should be displayed in the content of school curriculum, and basic functions of mathematics education should be implemented through the methodical system of teaching mathematics.

Modern scientists focus on the problem of subject-specialized studies in terms of high school. This problem has been which has been researched into by teachers and psychologists in its various aspects: the increasing interest of students to the knowledge of the chosen subject-specialization; the development and professional formation of personality traits (intelligence, communication, creativity); the students' focus on preparation to their future profession; focus of the subject-specialized studies on building the student-centered educational environment within which the student builds their own trajectory of education in socio-cultural space.

---

[1] *State national program "Education. Ukraine XXI century".* (1994). Kyiv: Raduha. (In Ukr.).

[2] *Law of Ukraine "On Education".* http://zakon4.rada.gov.ua/laws/show/1060-12. (In Ukr.).

[3] *The Law of Ukraine "On General Secondary Education".* http://zakon2.rada.gov.ua/laws/show/651-14 (In Ukr.).

[4] Berezivska, L., Bibik, N. & Burda, M. et al. (2003). *The concept of subject-specialized education in high school* [Approved by committee decision of the Ministry of Education and Science of Ukraine on 25.09.03 № 10 / 12-2] / Academy of Pedagogical Sciences of Ukraine. The Institute of pedagogy. *(Inform. Coll. Of Ministry of Education and Science of Ukraine),* 24, 3-15. (In Ukr.).

[5] *National Doctrine of the Development of Education.* (2002). *Osvita Ukrainy, 33.* (In Ukr.).

[6] *National Strategy for Development of Education in Ukraine for the period until 2021, approved by Decree of the President of Ukraine of 25.06.2013 number 344.* http://zakon4.rada.gov.ua/laws/show/344/2013. (In Ukr.).

Methodological aspects of subject-specialized studies in specific subjects (mathematics, computer science, physics, medicine) have been considered in the researches of T. Hordienko[7] (1998), L. Zhovtan[8] (2000), M. Pryhodii[9] (1999). Subject-specialized education organization of German high school students and primary school studies in Austria has been analyzed by M. Avramenko[10] (2007), and L. Fanninher[11] (2008).

PhD theses by O. Volianska[12] (1999) and M. Pischalkovska[13] (2007) focus on the problem of high school students' subject-specialized education in the system of educational institutions in the regional conditions. The concept of subject-specialized education in the rural secondary schools is grounded in the doctoral dissertation by N. Shyian[14] (2005).

Despite the diversity of scientific studies it should be noted that, the accumulated experience of preparing students for the conscious career choices, along with the undeniable evidence of its usefulness, reveals the minor efficiency of the profession-oriented educational activities in terms of subject-specialized training. The problem of high school students' preparation for the profession in terms of subject-specialized teaching of mathematics has not been particularly elaborated in the aforementioned works; the role of mathematical culture of specialists' professional development in the non-mathematical disciplines has not been fully reflected; the essential subject-specialization features and their relevant criteria availability, which would provide for the classification of areas in society on the grounds that determine what activities can be modeled in studies on a particular subject-specialization, have not been defined. The problem of subject-specialized education for the development of specific professional skills has not become the subject of special studies either.

All of the above mentioned has led to the need of developing the conceptual foundations for students' mathematical training in subject-specialized schools.

The research objective is to provide scientific grounds to the concept of high school students' mathematical education in a subject-specialized school.

The improvement of high school students' mathematical education in a subject-specialized school is based on the idea of creating a methodological system of high school students' profession-oriented mathematics teaching. The design and organization of the profession-oriented teaching mathematics in specialized schools should be based on the conceptual points:

1. Improvement of students' mathematical education in specialized schools should be based on a comprehensive system analysis of methodology with traditional and innovative ideas, approaches, and principles, along with historical and current trends of

---

[7]Gordienko, T. (1998). *Profile differentiation of teaching physics in the 10-11 forms of a secondary school (humanities profile)* (PhD thesis). The Institute of Pedagogy of APS Academy of Pedagogical Science of Ukraine, Kyiv. (In Ukr.).

[8]Zhovtan, L. (2000). *The differentiation in training the 7-11th grades in the process of in-depth study of subjects of natural-mathematical cycle* (PhD thesis). Lugansk state pedagogical university them. Taras Shevchenko, Lugansk. (In Ukr.).

[9]Prigodey, N. (1999). *Electrical Engineering Bias and Primary Professional Education in secondary school* (PhD thesis). Chernihiv state pedagogical university them. Taras Shevchenko, Chernihiv. (In Ukr.).

[10]Avramenko, M. (2007). *Profile Education in General School in the Federal Republic of Germany* (PhD thesis). The Institute of Pedagogy of APS Academy of Pedagogical Science of Ukraine, Kyiv. (In Ukr.).

[11]Fanninger, L. (2008). *Peculiarities of the profile-oriented education in secondary schools in Austria* (PhD thesis). Ternopol National W. Gnatjuk Pedagogical University, Ternopol. (In Ukr.).

[12]Volyanskaya, O. (1999). *Algebra and elementary analysis in the professional-technical school under the educational standard implementation* (PhD thesis). National pedagogical University named by M.P. Drahomanov, Kyiv. (In Ukr.).

[13]Pishchalkovska, M. (2007). *Profile education of the school-leavers* (PhD thesis). Institute of Pedagogical Education and Adult Education of the Academy of Pedagogical Sciences of Ukraine, Kyiv. (In Ukr.).

[14]Shyian, N. (2005). *Didactic Bases of Profile Education in Rural Schools* (Doctoral dissertation). Poltava State Pedagogical University them. V. Korolenko, Poltava. (In Ukr.).

the school mathematics education, advanced domestic and foreign experience of a subject-specialized school functioning[15] (Lovyanova, 2013).

2. Teaching mathematics in a subject-specialized school is seen as a system that combines appropriate goals, objectives, content, methods, forms, and tools and provides the expected students' learning outcomes which, in each sphere of subject-specialization (general cultural, applied, and theoretical), should be directed at receiving a high-quality mathematics education, the level of which is determined by the options of the subject-specialization training.

3. Designing the theory of students' profession-oriented mathematics teaching in subject-specialized schools should be carried out on the basis of defining the output parameters, definitions, theories, without which the key understanding of the professional orientation phenomenon in teaching mathematics is impossible, and the exploration of its functions in high school students' mathematical training as profession-oriented training, *firstly*, promotes the socialization of a high school student, psychological orientation to future profession, sustained interest in professional areas, and learning motivation which stimulates cognitive activity; *secondly*, it provides for the selection of educational content based on the interdisciplinary connections between specialized and general education and the elective courses[16] [17] (Lovyanova, 2013; Lovyanova, 2012).

4. The summary of the conceptual model of students' mathematical training in subject-specialized school is grounded on: the role of mathematical training in education; the group of principles, including: classical didactic principles; principles of subject-specialized education; principles of high school students' mathematical training; principles of designing the process of teaching mathematics in subject-specialized school; patterns of teaching mathematics. The basis of the conceptual model is the general scientific approaches (active, axiological, semiotic, competence-oriented, and system-structural) in the study and development of a student's personality in both – learning and teaching approaches (assignment approach) to implement professional orientation in teaching mathematics.

5. For the design and effective planning of students' mathematics teaching in subject-specialized schools to be proper, the use of subject-mathematical model of a subject-specialized school graduate's competency is crucial. Its component parts (targeting, motivational value, intellectual, cognitive, content and activity, organization and activity, control and reflective, and resulting) are consistent with the components of the structural and functional conceptual model of students' mathematical training in a subject-specialized school and determine the predictable result of profession-oriented mathematics training methodological system realization.

6. The methodological system of profession-oriented mathematics teaching in high subject-specialized school should: correlate the goals of the system with other external systems (education, operation of specialized schools, etc.) through external subsystems and determine the system of profession-oriented mathematics teaching through general-internal as well as internal subsystems specific purposes (*targeting system component*); provide for the formation of stable systemized knowledge of the science basics,

---

[15]Lovyanova, I. V. (2013). Retrospective analysis of the problems of teaching mathematics differentiation in comprehensive school. *Science and Education a New Dimension: Pedagogy and Psychology*, *5*, 114-119. http://scaspee.com/6/post/2013/07/retrospective-analysis-of-the-problem-of-differentiation-of-educating-to-mathematics-at-general-school-i-lovianova.html. (In Rus.).

[16]Lovyanova, I. V. (2013). On high school students' professional self-determination in the process of learning mathematics in a subject-specialized school. *Vesti BDPU*, *Issue 3, 4,* 35-38. (In Rus.).

[17]Lovyanova, I. V. (2012). Professional orientation in teaching high school students *Higher education of Ukraine: theoretical and methodical magazine*, *Special Issue "Pedagogy of high school: methodology, theory and technology"*, *3* (46), 170-180. (In Ukr.).

manifestation of the benefits of the structured subject teaching, creating optimal conditions for students' education, training and personal development towards professional orientation (*content system component*); combine the factors of a student's personality, interpersonal interaction such as *student-student*, *student-teacher*, psychological and pedagogical approaches to learning (*psychological system component*); provide learning and teaching activities aimed at a student's mastery of the mathematical educational activities (*activity and organization system component*); establish connections between the methods, techniques, organizational forms and teaching aids oriented towards professional orientation of a student's personality (*operational and technological component of the system*).

7. The design and implementation of the semantic content of the process of the development of high school student's professional orientation while teaching and learning mathematics should be taken into account as the variability of teaching mathematics in classes of different subject-specialization, features of the educational process in high school secondary education, psychological and educational characteristics of students who choose learning different subject-specializations and high school students' specific processes of socialization, self-development and self-realization will design a methodological system of profession-oriented teaching mathematics to meet the real learning process in subject-specialized schools as building a student's individual educational trajectory of mathematical training in such types of schools.

The students' mathematical training in terms of building individual educational trajectory will be a frame model with the following basic components: subject-specialization training, elective courses, and the academic level of the educational content[18] (Lovyanova, 2013).

The leading conceptual ideas allow to predict the following: the connection of goals and objectives of modern mathematical education with the students' personal development, the development of the professional orientation of a high school student's personality in the mathematics class will be facilitated with the introduction of the methodological system of teaching high school mathematics, which, dialectically combining purpose, content, methods, forms, and teaching aids are aimed at the formation of specialized school graduates' professionally important qualities; it takes into account the physiological characteristics of high school students' age; it is to provide for the psychological and pedagogical approaches to high school students' educational and personal development adapted to the information society.

That, in its turn, represents some aspects of students' professional orientation for specific educational subject-specialization. So, teaching mathematics in a subject-specialized school will contribute to the process of targeting students for their future profession, improving the quality of mastering mathematical knowledge, skills and abilities and, consequently, the components of professionally important qualities for a personality and a graduate's good mathematical education in specialized schools if: 1) mathematics teaching is provided in terms of achieving sustainable positive attitude to the subject in the specialized classes, revealing students' individual capabilities in the context of the selected subject-specialized training, formation of mathematical culture; 2) the classes of applied mathematics specialization are aimed at: gaining knowledge of mathematics for understanding the importance of scientific and technological progress, treating mathematics as a universal language of science, as a means of modeling phenomena and processes in nature and society, the development of logical thinking,

[18]Lovyanova, I. (2013). On the Specific Character of Mathematical Education Content Selection in a Subject-Specialized School. *American Journal of Educational Research, 1*(11), 523-527. http://pubs.sciepub.com/education-1-11-11.

algorithmic culture, and students' spatial imagination; 3) specialized math education is implemented under the following conditions: through identifying students' aptitudes to jobs directly related to mathematics; forming professionally important personality traits that promote optimal ratio between fundamentality and professional orientation of mathematical preparation.

The conceptual model of mathematical preparation in a subject-specialized school is built on the foundations of complex systems analysis, which allows to identify: conceptually important elements of high school students' mathematical training; the nature of their interaction, which ensures the integrity of the high school students' mathematical training; the function of each system component and high school students' mathematical training in general. The structure of the conceptual model of mathematical preparation in a subject-specialized school based on component analysis (detects system-forming and system-determining its conceptually significant elements), on structural one (grounds interaction character, relationships and connections between important conceptual elements of the system), on functional (clarifies the purpose of each component and the whole system) analysis allowed the study to characterize the integrity of the educational system, considering the complexity of its components, to describe the mechanisms and factors that provide such integrity, to find various types of bonds and synthesize them into a single theoretical picture. The outlined approach enables the formulation of the Concept of teaching mathematics in subject-specialized school concerning: the role of mathematical training in education, development and education of the personality; basic foundations of mathematical training model in a high subject-specialized school; key ideas of building a methodological system of profession-oriented teaching mathematics in a high subject-specialized school; terms of quality of specialized schools students' mathematical preparation.

Grounding on the patterns of teaching, principles of mathematics teaching design process in a subject-specialized school, psycho-pedagogical prerequisites for teaching high school students, the structural-functional model of students' mathematical preparation in subject-specialized school (Fig. 1) provides achieving the following results – forming an educated subject-specialized school leaver with the appropriate level of mathematical education.

The purpose and objectives of teaching mathematics, arranging a methodological system of professionally directed mathematics teaching and complex multi-level evaluation of the results of mathematical education in subject-specialized schools are within the model in constant interdependence, they are consistent with the school-leaver's subject-mathematical competency model and outline the result - the formation of an educated subject-specialized high school leaver with the appropriate level of mathematical education corresponding to certain educational subject-specialization. The result is achieved by organizing students' EMA at all stages of the model organization.

In the research the students' educational mathematical activity is interpreted as an active teaching and learning activities within the chosen level of mathematical training aimed at learning the subject, due to the possibilities of teaching mathematics at in specialized schools – from the different levels of mathematical training mathematics at different levels of mathematical training in specialized schools – from the development of elementary and basic skills at the standard level to mastering the methods of mathematical modeling at the academic level and the elements of creativity inherent to professional mathematician at the subject-specialized level. It has been proved that the formation of educational mathematical activity is fostered by the assignment approach to teaching mathematics, which provides the system of professionally designed tasks in the scope of mathematics teaching content.

Complex multi-level result evaluation of high school students' mathematical preparation is enabled by the characteristics of motivation and value component, information

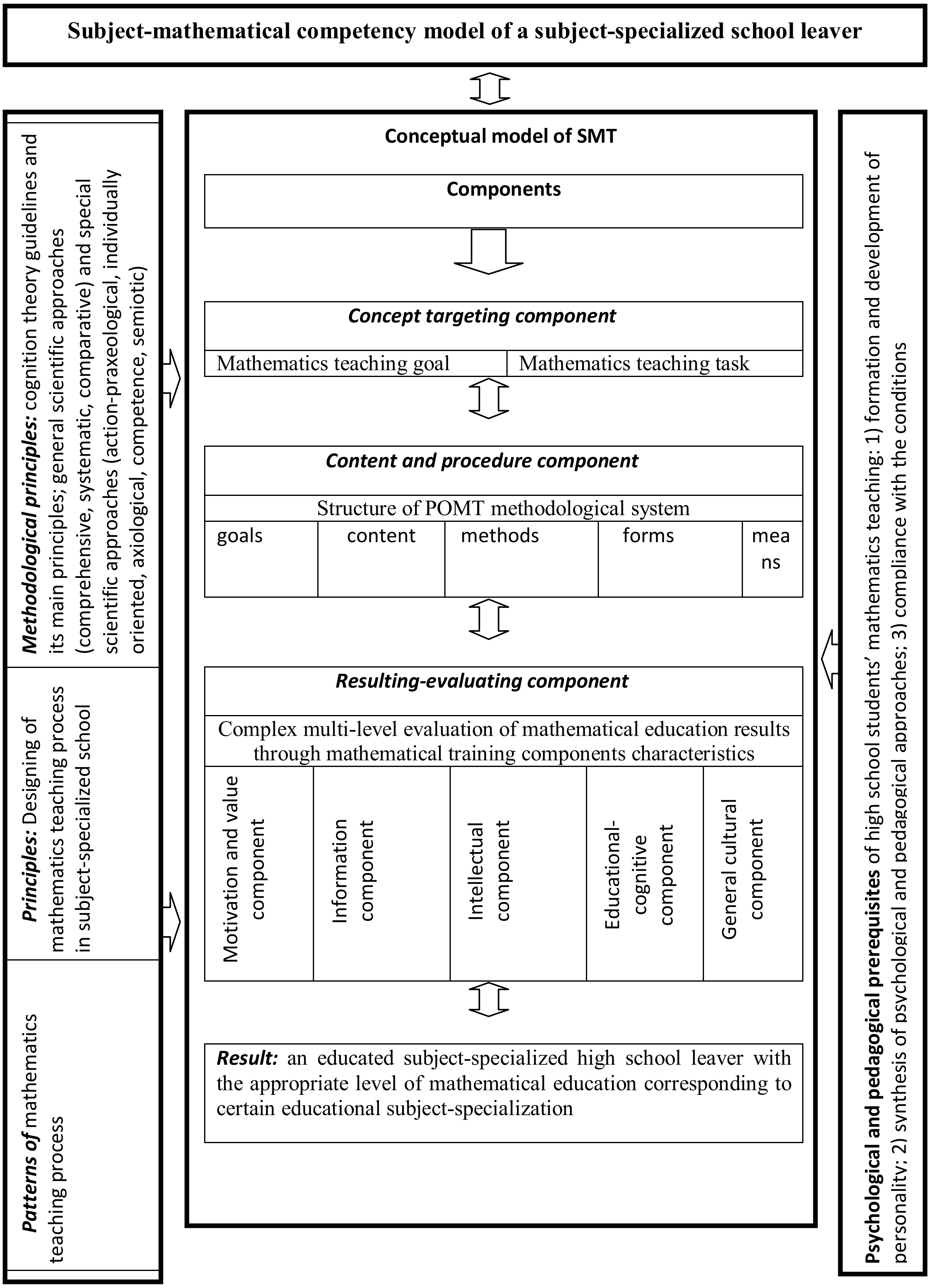


Figure 1. Structural-functional model of high school students' mathematical training in a subject-specialized school

component, intellectual component, educational and cognitive components of general cultural mathematical training.

The results of the study can be used in the mathematics teaching practice in schools of secondary education; during the development of learning and teaching aids, in-service training for teachers; training Physics and Mathematics students in classical and pedagogical universities.

The given research opens a new trend in the theory and methods of teaching mathematics, involved in the solution of professional orientation problems in teaching mathematics to the subject-specialized schools' students. A further development of theoretical foundations of teaching mathematics to high school students is methodologically motivated by the need to create innovative teaching-methodological aids in terms of a subject-specialized school, to analyze the problems of continuity and succession in the implementation of the system "subject-specialized school – university".

4.3. Research Approach in the Current School Mathematical Practice

L. Golodiuk

Introduction. Updating content of teaching mathematics and teaching technology, harmonizing them with current and future development of the society, integration to the European educational environment is the consideration of the learning process and its results through the prism of competence-based approach, which can be interpreted as the factor of modernization of the content of mathematical education and its complements through a number of educational innovations and classic approaches with the aim of achieving current educational goals of teaching mathematics, in particular the formation of the subject mathematical competence of a student.

The approval of the new edition of the State standard of basic and complete general secondary education led to the modernization and improvement of content of a primary school mathematics curriculum, and transferred pedagogical research into the plane of change of the methodological approaches to teaching mathematics.

Determining the facilitating conditions for the formation of mathematical competence, we draw attention to the practical orientation of learning, which provides continuous involvement of students in different types of pedagogically appropriate active educational activities, namely, in learning and research activity, as well as its practical orientation, providing the formation and development of learning and research skills.

The allocation of learning and research activity of students among other types of educational-cognitive activities and the emphasis on the systemic and systematic organization is the need of the day. This activity type is considered, on the one hand, as an effective means of formation of objective and crucial mathematical competence, and on the other hand – as a means of personal development of a student.

Arrangement of educational activity is based on students' needs to carry out a conscious transformation of mathematical learning material with the aim of mastering new knowledge and ways of activity. Regarding the organizing of educational and research activity of students at mathematics lessons, we approach the formation and development of learning research skills of students as its basis. Among the main components of learning and research skills of students in mathematics, we allocate such skills: social and interactive; intellectual and creative; perceptual and informational; reflective and analytical; organizational and adaptive.

The process of the formation and development of students' learning and research skills is most intense while performing different mathematical tasks united in a system (learning and research tasks) created on the basis of system-structural approach.

New knowledge, which the child masters as a result of the implementation of learning and research activity, acts as a direct product of knowledge.

Analysis of recent research and publications showed that the problem of learning and research activities of students is the subject of scientific analysis of domestic and foreign scholars whose research is a complicated branching of scientific approaches to this issue and the ambiguity of the scientists' conclusions. S. Rubinstein[1] (1973), O. Leontiev[2] (1975), V. Shtoff[3] (1963), N. Talyzina[4] (2007) and others observe various aspects of cognitive activity in their works. The research activity of students has been the object of study of such scholars as V. Palamarchuck[5] (1987), O. Savenkova[6] (2004),

---

[1]Rubinstein, S. L. (1973). *Fundamentals of General Psychology*. Moscow: Pedagogika. (in Rus.).
[2]Leontiev, A. N. (1975). *Activity. Consciousness. Personality*. Moscow: Politizdat. (in Rus.).
[3]Shtoff, V. A. (1963). *The role models in cognition*. Leningrad: Izdatelstvo leningradskogo universiteta. (in Rus.).
[4]Talyzina, N. F. (2007). The essence of the activity approach in psychology. *Methodology and History of Psychology*, 4, 157-162.
[5]Palamarchuck, V. F. (1987). *School teaches us to think*. Moscow: Prosveshchenie. (in Rus.).

O. Savchenko[7] (2012) and others. The works of V. Andreiev[8] (1981), V. Guzeiev[9] (2002), I. Lerner[10] (1970), M. Skatkin[11] (1986) and others focus on the development of new directions of learning and learning activities. Studies focused to the psychological characteristics of teaching pupils and students, teaching skills and patterns of skills are performed by V. Davydov[12] (1995), M. Makhmutov[13] (1975), I. Kharlamov[14] (1997) etc.

Development of theoretical and methodological aspects of teaching mathematics is reflected in the writings on the methods of forming mathematical knowledge[15] [16] (Sliepkan, 2004; Tarasenkova, 2013).

However, without denying the significant contribution to the solution of this problem made by the above-mentioned authors, it should be noted that the process of organization of learning and research activity of students at mathematics lessons requires methodological and technological clarifications. To define the innovative trajectory of methodological system of the renewal of teaching mathematics, we have organized and carried out a survey of teachers. The questions of the survey allowed us to determine the level of understanding of an actual problem in the formation of teaching and research skills of students in learning mathematics. Let us discuss some issues where the answers require scientific theoretical and practical rationale.

Question 1. "To your mind, what activities, in which a secondary school students are involved, can be called teaching and research activities?". The respondents gave different formulations: activities related to solving the problematic learning situations (32%); project activities (30%); search activities (7%); the activities involving the use of information and communication technologies (9%); it is difficult to define (1%); there is no answer (21%).

There sure is an identification of the concept of "teaching and research activities" with the term "activities related to solving the problematic learning situations." Obviously, this fact requires a theoretical explanation and justification, because the teachers do not understand the conceptual apparatus of innovation that leads to the disruption of technology of its use.

Question 2. "How do you understand the meaning of the phrase "learning and research skills of mathematics students"?" For most teachers this issue turned out to be difficult, that is why 60% of respondents did not formulate the answer.

Question 3. "Which educational methods (based on educational-cognitive activity of students) do you apply for students’ learning and research activity?". The respondents gave different formulations: explanatory and illustrative (2%); reproductive (4%); problematic (43%); partially search or heuristic (21%); research (17%); active (13%).

Considering the results of the survey the aim of the study is the scientific interpretation of the basic concepts (activity, educational activity, learning and research activity, teaching methods) and exposing the methodological aspects of teaching mathematics on the basis of educational and research tasks.

---

[6]Savenkov, A. I. (2004). Research training and engineering in modern education. *Research work students*, 1, 22-32. (in Rus.).

[7]Savchenko, O. YA. (2012). Learning environment as a factor of stimulating the research activity of primary school children. *Scientific notes Minor Academy of Sciences of Ukraine*, 1, 41-49. (in Ukr.).

[8]Andreiev, V. I. (1981). *Heuristic programming of educational and research activities*. Moscow: Vyisshaya shkola. (in Rus.).

[9]Guzeiev, V. V. (2002). *Methods and organizational forms of learning*. Moscow: Narodnoe obrazovanie. (in Rus.).

[10]Lerner, I. YA. (1970). The construction of the logic of pedagogical research. *Soviet pedagogy*, 5, 3-10. (in Rus.).

[11]Skatkin, M. N. (1986). *Methodology and methods of educational research*. Moscow: Pedagogika. (in Rus.).

[12]Davydov, V. V. (1995). On the concept of developmental education. *Pedagogy*, 1, 29-32. (in Rus.).

[13]Makhmutov, M. I. (1975). *Problem teaching*. Moscow: Pedagogika. (in Rus.).

[14]Kharlamov, I. F. (1997). *Pedagogy*. Moscow: Yurist. (in Rus.).

[15]Sliepkan, Z. I. (2004). *Psycho-pedagogical and methodological principles of developing teaching mathematics*. Ternopil: Pidruchnyky i posibnyky. (in Ukr.).

[16]Tarasenkova, N. (2013). The quality of mathematical education in the context of Semiotics. *American Journal of Educational Research*, 1(11), 464-471.

The presentation of basic material. Theory of activity is seen as a system of methodological and theoretical principles of study of mental phenomena. The main object of study is the activity. This approach is examined on two levels: the principle of unity of consciousness and activity[17] (Rubinstein, 1973) and the problem of common structure of external and internal activities[18] (Leontiev, 1975).

In his writings, L. Rubinstein considers the activity as a set of actions aimed at achieving the objectives. According to L. Rubinstein, the main features of the activity are such features: sociality (activity is held only by a subject of activity); activity as the interaction of subject and object is meaningful, substantive; activity is always creative and independent. Activity is determined by its object not directly, but only indirectly through its internal specific patterns (due to goal, motivation, etc.). This is a partial manifestation of the general principle of determinism: the external forces act only through the internal conditions of at whom or at what these external influences are pointed. From these perspectives a theory of thinking is created as an activity and as a process.

According to O. Leontiev's theory, a human is described only by those mental processes and features that facilitate the implementation of a person's activities. The hierarchy of activities forms the core of the personality. The main characteristic of the individual is self-awareness that is the understanding of one' place and positioning themselves in the system of social relations. Each age period of a personal development, according to the theory of activity, corresponds to a certain type of activity that takes a leading role in the formation of new mental processes and traits of the personality.

G. Shchukina[19] (1963) considers activity as a major form of the manifestation of human activity and a human's social purpose. The essence of human activity is the transformation of the reality and active influence of the individual on the objective world. The scientist identifies the following main characteristics of a common phenomenon of activities:

1) goal-setting (transformation of general purpose into specific tasks);
2) being transformative in nature (activities with the prospect of improving the environment, transformation of the world);
3) objectivity (expressed its objective material basis, its relation to the objective world);
4) deliberate nature (reveals their subject, which is found in goal-setting, in forecasting activities, in promising aspirations)[19] (Shchukina, 1986).

So, we can say that not certain properties determine the feature of human activity, but their relationship makes the unity and integrity of any activity and its variability. During activities a child's develops in the objective world, and also forms the attitude to it, to one's own place in this world, to the society, to people with whom the child is learning.

The development of a problem of leadership activities became a fundamental contribution of O. Leontiev (1975) to the development of the age psychology. The scientist has not only described the change in leadership activities in the development of the child, but also initiated the study of the mechanisms of these changes, a transition of one leadership activity to another. Let's note that the leading activity (Leontiev, 1975) is an activity that has three characteristics:

1) activities, in the form of which other new types of activities occur and within which they differentiate;
2) activities where mental processes (thinking, perception, memory, etc.) are formed or developed;
3) activities which determine basic psychological personality changes in this period of development.

---

[17]Rubinstein, S. L. (1973). *Fundamentals of General Psychology*. Moscow: Pedagogika. (in Rus.).
[18]Leontiev, A. N. (1975). *Activity. Consciousness. Personality*. Moscow: Politizdat. (in Rus.).
[19]Shchukina, G. I. (1986). *Role activities in the learning process*. Moscow: Prosveschenie. (in Rus.).

D. Elkonin[20] (1971) believes that a leading activity in adolescence is communication which is based on different types of socially useful activities.

Recently the issues related to the specification of students' activity have been discussed in the literature; and a research activity is identified as one of the types.

According to M. Kniazian[21] (2007), the research activity is called one of the types of students' creative activity and is characterized by several features:

1. Research activities are related to students' solving of creative tasks.
2. Research activities should take place under the supervision of a specialist.
3. The main task of studies is to obtain new knowledge. The task should be feasible for students.
4. Research activities can engage all students: those who have a high level of training and those who have an average level.

In this description the features are listed so clearly that we can arrange them in a linear chain for cause-and-effect relations: *research activity – a kind of creative activity → solving creative problems | teacher-led | to acquire new knowledge |students of high and average levels.*

Another controversy is the generalization that in the process of research activities students can perform only creative tasks. Take into account the opinion of L. Shelestova[22] (2003), who defines the latter: "a creative task is a task during which a student creates a creative product", the organization of students' research activities is limited by the content of the tasks performed in the study of mathematics and the level of the students' achievements.

More comprehensive is the definition of "research activities" submitted by S. Serova and N. Fomina[23] (2006), including "Research activities are cognitive activities aimed at developing new knowledge about objects and processes, deepening the knowledge gained about the subject, the realization of their own desires and capabilities, the satisfaction of interests, the disclosure of instincts and abilities of each child. This activity involves obtaining by each participant a specific result in a set of knowledge and skills...".

We agree with the authors who distinguish the research activity as a component of cognitive activity that is directed at the formation of new and deepening the existing knowledge through interest and intrinsic motivation of students to the formation of knowledge and skills.

Without dwelling on the analysis of the nature of the other options of the definitions of a notion "the research activity" and taking into consideration its multidimensional nature, which became the basis for launching new research, particularly in the sphere of teaching and research, search, scientific and research work, let's single out for further disclosure the concept of "teaching and research activities".

According to the definition given by O. Obukhov[24] (2006), teaching and research activity of students is a creative process of common activities of the two entities (a teacher and a student) to discover the unknown, in the course of which the transmission between cultural values is carried out, which results in the formation of ideology. Describing the teaching and research activities, he emphasizes the basic function: the desire of students to the cognition of the world, of themselves, and of themselves in this world. We believe that the purpose of teaching and research activities of students is to

---

[20]Elkonin, D. B. (1971). On the problem of periodization of mental development in children. *Questions of psychology,* 4, 6-20. (in Rus.).

[21]Kniazian, M. O. (2007). Formation of self-research as a pedagogical problem in the theory and practice of preparing future teachers. *Problems of Education,* 51, 92-98. (in Ukr.).

[22]Shelestova, L. (Ed.). (2003). *How to help your child become a creative person.* Kyiv: School world. (in Ukr.).

[23]Serova, S. O., & Fomina, N. V. (2006). Way into the world of scientific technology. *School management*, 3, 27-29. (in Ukr.).

[24]Obukhov, A. S. (2006). *The development of research activity of pupils*. Moscow: Prometey, MPGU. (in Rus.).

provide a focused personal development, acquisition of research skills, mastery of knowledge that is perceived as new and personally meaningful only in relation to specific individual. The object of these activities can be an educational and research objective, which in its essence is informative and oriented at the "zone of proximal development" of the child. Teacher's training of the formation of the students' teaching and research skills is preceded by a successful, effective implementation of teaching and research activity of students.

In organization of scientific and research activities of students several principles should be observed[25] (Byelich, 2012):

– students' research activities are close to the scientific and research activities, they further develop in scientific activities;
– a content of the study should be combined with an educational purpose, general public needs, and issues of today;
– scientific research is a continuous process, it cannot be run in a few days;
– scientific and research activity is always a controlled process.

Such work of students must meet the scientific methods of cognition, enhance the content of their education and improve the preparation to future activities.

For organizing students' learning and research activity the teacher should clearly understand what teaching methods should be used in the variety of learning situations.

The works of many scientists focus on the classification of teaching methods, where a method as a component of the study is investigated from different perspectives: historical, psychological, epistemological, logical, and methodical.

In pedagogy, particular in didactics, a teaching method is seen as a way to achieve the learning objectives, it's a tool of educating and developing the students in the process of common activity with a teacher; a method of teaching activity and the organization of learning and cognitive activity of students in solving various didactic problems is aimed at mastering the material being studied; an interdependent set of thinking actions; ways of solving learning tasks, all of which are specific learning objectives, etc.

The complexity of this phenomenon, hence the possibility of different approaches to the disclosure of its essence, explains the simultaneous existence of different concepts and classifications of teaching methods:

– acquiring knowledge, methods of formation of knowledge, skills and abilities, methods of application of received knowledge, skills and abilities[26] (According to Danylov & Yesypov, 1957). Classification features: The main teaching tasks that are needed to be addressed at a particular stage of learning;
– verbal explanation, story, lecture, discussion, work with a textbook; visual illustration, demonstration, independent observation; exercises, laboratory works, practical, graphic and research works[27] (According to Babanskyi, 1985). Classification features: Organization and implementation of learning and cognitive activity;
– explanatory and illustrative (informational and receptive); reproductive, problem presentation; partial search (heuristic); research methods[28] (According to Lerner & Skatkin, 1982). Classification features: The content of education and ways of learning is the type (character) of learning and cognitive activity.

In our study, we adhere to the classification developed (by Lerner & Skatkin, 1982), noting that the first two methods are aimed at organizing students' reproductive activity

[25]Byelich, N. I. (2012). Involvement of students in research work. *Retrieved* July 02, 2016. http://osvita.ua/school/lessons_summary/upbring/27192/. (in Ukr.).

[26]Danilov, M. A., & Esipov, B. P. (1957). *Didaktika.* Moscow: Uchpedgiz. (in Rus.).

[27]Babanskij, YU. K. (1985). *Methods of teaching in a modern comprehensive schoo*l. Moscow: Prosveshchenie. (in Rus.).

[28]Skatkina, M. N. (Ed.). (1982). *Didactics High School: Some problems of modern didactics* (2th ed.). Moscow: Prosveshchenie. (in Rus.).

and the last three ones are aimed at productive, creative activity.

Using the productive teaching methods, based on the determining factor that is the nature of student's learning and cognitive activity, is aimed at changing the nature of students' activity from performing, active performing, active independent activity to creative independent activity and as a result a progressive personal development: changing student's positions from performing $\rightarrow$ to active $\rightarrow$ to the position of the subject.

Problem method of teaching mathematics, the essence of which is to ensure the effective ratio of students to form the key competencies, intensive development of self-learning and cognitive activity, and individual creativity in the implementation of educational objectives and tasks, is based on the implementation of interactive learning. The interactive learning is seen as "a form of learning and cognitive activity of the student, which is aimed directly at their participation in the verbal filling of local educational and information space, which serves as the topic of the lesson"[29] (Chernets'ka, 2011).

We determine problem questions as the basic units of the problem method implementation.

A problem question is a question without any prompt about the answer; the students find the answers themselves. Unlike usual questions, a problem question does not provide a simple guessing or reproduction of information, the answer to this question requires performing simple productive mental operations. In practice, the use of problem questions is effective after learning new material (mathematical concepts), for example: Is there a divide for the prime multipliers equality $5 = 1 \cdot 5$?; Can we consider that the numbers that are divided into 9, can also be divided into 3?[30] (Tarasenkova, 2014; & others).

The result of the psychological research into the conditions of raising and asking problem questions at different age stages (pre-primary age – 5-6 and 6-7 years old; primary school age – 8-10 years old; adolescence – 11-15 years old) was the discovery of two sensitive periods in the development of questions[31] (Shumakova, 1986). In the first period (5 to 7 years) there is a sharp jump in the level of search activity in the form of questions. During these years, the child has a free form of research-oriented questions. In this period a child's questions are projected from questions directed by adults to the problem questions aimed at independent, searching disclosure of the unknown.

At the beginning of adolescence (11 years old) the growth of search activity in the form of questions ceases. Adolescence (11-15 years old) is the second psychologically favorable period in the development of questions. A teenager moves from the general problem situation to a deep examination of the problem – selective examination. Having found out the problem (unknown), a student pauses upon comprehensive examination, trying to find answers to questions on their own. This period of a child's development should be used to enhance the effective influence on the level of cognitive activity and creativity of a learner.

From the above we can conclude that the problem method is an effective method of teaching mathematics, when students are in the 5-6th grades; we define it as a basis for the introduction of partial-search method of teaching in grades 6-8.

A partial-search method of teaching is based on the organization of students' participation in the implementation of some steps in the search of a problem solution. The role of the teacher is to create learning and cognitive tasks, arrange them into steps, and identify those steps that students will perform on their own that is the teacher

[29] Chernets'ka, T. I. (2011). *Modern lesson: theory and practice of modeling*. Kyiv: TOV "Praymdruk". (in Ukr.).

[30] Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M., & Serdiuk, Z. O. (2014). *Mathematics*: *textbook for the 6th form for the secondary schools.* Kyiv: Publishing House "Osvita". (in Ukr.).

[31] Shumakova, N. B. (1986). Research activity in the form of questions in different age periods. *Voprosy psihologii*, 1, 27-32. (in Rus.).

organizes partially independent learning and cognitive activity of students based on interactive learning.

"Interactive learning is seen as a form of learning and cognitive activity of the student, the implementation of which provides simultaneous execution of the same educational and cognitive tasks, characterized by problematic, informative, developing or controlling overtones, focus on the direct or indirect involvement of students in the verbal content of the local educational and information space of a lesson, which serves as the topic of the lesson and is organized in accordance with one of the three forms of interactions: co-individual; co-coherent; co-interacting"[32] (Chernets'ka, 2011).

The main component of interactive learning is communication. It is considered as a multi-faceted process of establishment and development of contacts between subjects, that involves the exchange of information, certain tactics and strategies of interaction, perception and understanding of each other during communication. Cognitive communication predominates in the learning process, which we consider as communication that implements all the functions of knowledge on a particular subject material.

Isolating the basic unit of introducing partial-search method – a problem situation which is described as an intellectual difficulty that arises when a student does not know how to explain certain facts or phenomena and cannot achieve the desired goal by already known ways, we consider some examples (table 1).

Table 1. Examples of educational problem situations

Topic, grade	Problem situation	Methodical comments
The sum of angles of a triangle	Construct a triangle with the degree measure of the interior angles respectively equal to: 50^0, 70^0, 80^0	The sum of measures of the interior angles of a triangle is equal to 180^0. Therefore, the triangle with given data cannot be built
Inequality of a triangle	Construct a triangle with the sides length equal to 5 cm, 7 cm and 8 cm	The triangle can be constructed if the length of a side of a triangle is not bigger than the sum of the other two sides of the triangle

The process of management students' solving a problem situation has four components:

1. Preparing for solving a problem situation (formulation of a hypothesis (assumption); setting a learning and cognitive task; an indication of the sequence of activity; distribution of differentiated didactic material).
2. Organization of work (reading the content of the materials and planning; dividing the task into some parts; individual performance of the task; discussing the results; additions, clarifying, summarizing the task; summing up the results).
3. Unaided work of students with the problem situation (communication in a system teacher-student, student-teacher, student-student, student-computer, etc; work on the instructions, solving using the algorithm, etc.).
4. Finishing the work (general conclusion about the achievement of the goals and refutation or confirmation of the hypothesis (assumption)).

In the process, the teacher performs various functions: monitors the progress, answers questions, provides individual assistance to students, and creates a favorable psychological background, so we organize the type of communication the structure of which is classified by L. Friedman[33] (1987). Communication consists of three interrelated components: 1) communication (information exchange between people in

[32]Chernets'ka, T. I. (2011). *Modern lesson: theory and practice of modeling*. Kyiv: TOV "Praymdruk". (in Ukr.).

[33]Fridman, L. M. (1987). *Teaching experience through the eyes of a psychologist*. Moscow: Prosveshchenie. (in Rus.).

the process of communication); 2) interactive (the interaction between individuals); 3) perceptive (the process of mutual perception of partners in communication and the establishment of the basis of emotional relationship to each other). While talking, discussing and solving problem situation, we formulate questions that are aimed at creating common methods of analysis and solving the whole class of problems. For example, the following questions can be asked: What are the processes described in the problem? What quantities is each process characterized by? What is known about each quantity? Can you find a relationship between the quantities?

Such questions will organize the work of students in the main phase of solving – analysis of the situation. Questions that form a system of benchmarks can be used in the study of the problem situation. This changes a students' approach to the study of theoretical material. The theory is seen not only as an object that should be memorized, but also as a basis for practical application. Questions help to understand the essence of the problem situation, to establish a new relationship with the material previously studied.

For example: Based on life and learning experiences, set a formula of square shapes shown below (table 2).

The research method of teaching mathematics lies in the introduction of general and partial methods of scientific research in the learning process at all stages (from perception to use in practice); organization of activities at the lesson and after lessons; updating interdisciplinary connections; complication of content and improvement of procedural and substantive aspects of learning and cognitive activity. There is a change of character of relations "teacher - student – group of students" in the direction of cooperation. A method of learning: interactive learning interchange (mutual learning) and internal learning (I teach myself).

"The internal learning is considered as an individual form of learning and cognitive activity, which includes independent accomplishment of the task, part of the task, exercises, etc"[34] (Chernets'ka, 2011).

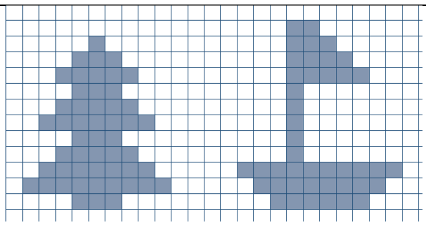
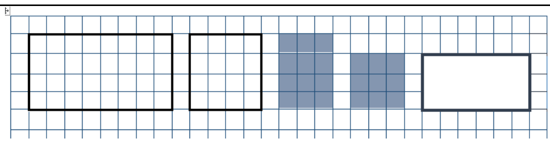
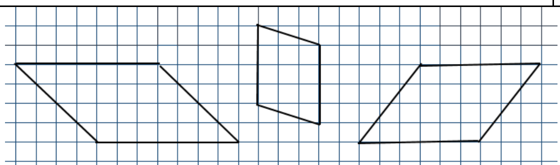
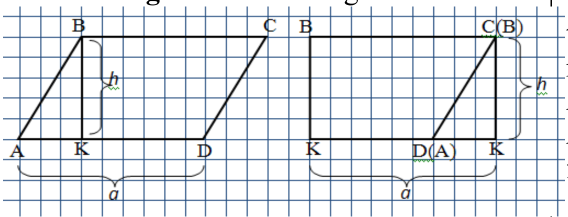
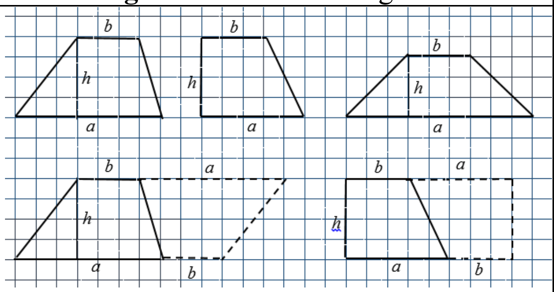
The basic unit of the implementation of the research method is the teaching and research task. These tasks are closely linked to content (theoretical) generalization, they lead the student to the formation of skills and abilities to summarize and organize training material, to learn new ways of action and formation of learning experience.

In the process of performing learning and research tasks students acquire learning and research skills, namely: intellectual and creative (skills that ensure the effectiveness of implementation of mental operations of comparison, analysis, synthesis, generalization, classification and provide effective mental activity); social-interactive (skills, which are based on actions aimed at establishing and maintaining effective interaction between actors); perceptual and informational (skills that are supported by actions of active perception, memorizing, retention, reproduction and structuring of information; found in the implementation of an effective process of information perception and manipulation of its contents); organizational and adaptive (skills, that provide a child's productive entering the information-educational environment, occur through planning the unaided activity in accordance with its purpose, presuppose the choice of ways of achieving the objective and the necessary funds, determine the sequence of actions in the structure of the activity); reflective and analytical (skills that are carried out using the procedure of introspection as a process of obtaining a certain result and self-regulation as a process of self-formulation of the objective of activity and to ensure its implementation).

The learning and research tasks allow us to develop learning motivation; to stimulate the mechanisms for student's orientation; to provide an independent goal setting for future learning activity; to form general educational and special skills of students;

[34] Chernets'ka, T. I. (2011). *Modern lesson: theory and practice of modeling*. Kyiv: TOV "Praymdruk". (in Ukr.).

Table 2. Tips for setting formulas of square shapes (for students of all ages)

Educational tasks	Methodical comments
 **Figure 1.** Shape	At primary school students are taught to identify shapes using a square cell. To calculate the area of data pieces, you need to count the number of cells and multiply by 0.25 cm^2. The total formula of finding area: S = 0,25n, where n – number of cells that figures are limited by
 **Figure 2.** Quadrangles	At primary school students learned the formula of the area of a rectangle and a square. Specifically, S=*ab* and S=*aa*=a^2. Thus, to calculate these figures it is enough to find the lengths of sides of a rectangle and a square
 **Figure 3.** Parallelograms  **Figure 4.** Conversion figures	Calculating the area of a parallelogram can be possible if we use the recommendations of the first task. But to apply this method to determine the area of the second parallelogram is impossible. We recommend to carry out a series of reformations: lower the BK height; "cut" the ABK triangle and move it, so that the CD side coincides with AB side, AC side with the continuation of AD side; as a result of changes rectangle is formed the area of which we can find. S=*ah*
 **Figure 5.** Trapezium. Its transformation	To calculate the area of a trapezoid we use completion elements. We extend a smaller trapezoid basis of the length of each corresponding larger base. We obtain a parallelogram, which consists of two equal trapezoids. S=(*a*+*b*)*h*

to strengthen the moral-volitional and physical quality of the educational goals of the student on the achievement of results; to maintain the learning capacity of the child; to provide self-esteem of activity; to create conditions for the manifestation of the supreme personal functions.

We have identified such components of the learning and research task focused on the formation of the objective mathematical competence of students in adolescence:

– tasks of the formation of classification and summarizing charts, tables;
– tasks of the actualization of methods or ways of solution;
– tasks of the selection of the generalized algorithm, way or method of solution;
– tasks of establishing the properties of figures;
– tasks of the study of properties of geometric configurations;
– tasks, which are based on mathematical description of various real processes and situations;
– tasks of the development of algorithmic and heuristic advice;
– tasks of the modeling;

– tasks of summarizing the conclusions that can be used to solve personally meaningful problems.

The study of mathematics by younger teenagers is carried out with a predominance of inductive reasoning mainly on the visual-intuitive level with the practical experience of students and examples from the environment. There is a gradual increase of the theoretical material that requires the substantiation of statements that are studied. It prepares students to use deductive methods in the next stage of learning mathematics. Special attention should be paid to the geometric material (measurement of geometrical quantities and the construction of geometric figures) that has a practical application. While studying this material we need to form the following practical skills: measure a segment and build a segment of a given length, measure the angle with the given degree measure, measure the angles of a triangle and build the triangle with the given sides and angles (simple cases), measure and find the area of a rectangle, square, and the size of a rectangular parallelepiped and a cube.

The study of the definitions of triangle and rectangles as particular types of polygons provides the basis for the propaedeutics of elements of deduction and also contributes to systematizing knowledge about geometric shapes. Polygons as angles are considered together with the internal area which gives the opportunity to divide the angle into parts and determine the area of a polygon. The introduction of such concepts as pyramid, sphere, cylinder and etc. gives us the opportunity to expand understanding of the spatial figures.

Thus, the study of the geometry is associated with the measurement and calculation of size, allowing us to illustrate the spatial and quantitative characteristics of real objects, to organize productive activities of students. In the process of learning geometric material: the development of students' spatial concepts, the abilities to observe, compare, generalize and to abstract occur; practical skills to build, draw, model and construct geometric shapes by hand and using simple drawing tools are formed; notions and concepts of geometric figures in the plane and their essential characteristics and properties are formed; learning to recognize geometric figures in space and their elements, a comparison of images of geometric shapes with the surrounding objects; formation of skills to determine space of geometric figures.

Systematizing learning and research tasks the teacher follows the general sequence of elements of study:

1) the subject;
2) the objective;
3) the hypothesis / assumption;
4) tasks;
5) the procedure;
6) conclusions.

The subject of the learning and research tasks is determined by the teacher in accordance with the curriculum. Formulating the purpose of the learning and research task, students learn to predict the result of the study, that is, what in general must be achieved in the end. The objective should be formulated concisely and as clearly as possible to express the main thing that should be achieved in the process of performing learning and research tasks.

The objective is formulated though target actions, in this case it is mandatory the use of such verbs as: to develop; to justify; to characterize; to identify; to determine; to check; to disclose; to study; to provide; to explore etc.

In the formulation of the objective, the target object is determined, which is subjected to direct examining in the framework of the task, outlining the system and students' activities.

The target of learning and research task is a part of theory and practice, within which the achievement of the objective is implemented. In these learning and research tasks it is the geometric concept or geometric shape.

Requirements for the formulation of objectives for learning and research tasks:

1) be achievable;
2) be defined by topic and problem;
3) the ability to check the objective (controllability, diagnostics of the object);
4) to be unambiguous and to present the final result of the target action as a summary.

The term "hypothesis" is interpreted in the Large explanatory dictionary of the modern Ukrainian language in two meanings: a scientific position, a conclusion that explains certain phenomena of reality on the basis of assumptions; any assumption, conjecture[35] (Byelich, 2012).

Thus, under the hypothesis we understand the assumption itself, which explains the observed phenomenon, and the way of thinking in general, which involves the extension of the assumption, its development and substantiation. A hypothesis is a method of cognition of objects and phenomena of the surrounding world.

The hypothesis can explain the phenomenon (event) in general, or some particular part of the phenomenon, one property, one relationship. Therefore, we will distinguish between general hypothesis and partial hypothesis.

Under the general hypothesis we understand the hypothesis that explains the cause of the phenomenon or groups of phenomena in general.

A partial hypothesis is a hypothesis which explains a certain part or a certain property of the phenomena, event.

For example, the hypothesis of polygons is the general hypothesis, but the hypothesis of the square is the partial hypothesis.

Under the hypothesis of learning and research task we will understand the partial educational hypothesis, which is formulated for a specific task. This will allow the student to conduct educational research and to confirm or refute (to adjust) available learning experience.

The hypothesis is probabilistic and during its formulation we must take into account three stages:

1) the level of formation of mathematical competence of a student;
2) formation of a hypothesis and its substantiation;
3) verification of the received results in practice.

The formulation of a hypothesis of learning and research task allows us:

- to highlight simple and accessible operations for performing;
- to determine the sequence of these operations, in accordance to their connection, difficulty and execution time (to develop tactics of research);
- to build the entire amount of work on learning and research task.

The selection and ordering of learning and research task should be clear enough, as a rule, it is the enumeration of what must be done in accordance with the objective of learning and research task, to confirm the given hypothesis.

A clear formulation of the tasks allows us to organize the process of learning search, to systematize the research process, to increase its efficiency. The sequence and content of tasks are formed according to the purpose and hypotheses of learning and research tasks, which gives the research process straightforward.

Examples of statement of objective, hypotheses, tasks of the learning and research task in mathematics for students in adolescence.

In primary school students learn to mark points, draw segments, and name them correctly. They understand that through the two points they can draw only one line and that the lines cross at one point.

[35] Byelich, N. I. (2012). Involvement of students in research work. *Retrieved* July 02, 2016. http://osvita.ua/school/lessons_summary/upbring/27192/. (in Ukr.).

Table 3. The subject of the study. Point, segment, ray, line, plane

Elementary school	Point	• – this is the point.
	Line	Line without beginning or end.
	Ray	Part of the line, which is only bounded by a point only on one side.
	Segment	Part of the line, which is bounded by points on both sides. The points are the ends of the segment.
Secondary school	Point	The point is the simplest geometric figure. To mark a point, it is enough to touch the paper with a pen.
	Line	The geometric figure "line" is infinite.

The purpose of the study. To study the point, segment, ray, line, plane, and to establish which of these shapes are the simplest geometric shapes (table 4).

Table 4. The example of formulating a hypothesis of the learning and research task

The structure of formulating hypothesis of the learning and research task	**The example of formulating hypothesis of the learning and research task**
The hypothesis of the study. I think the simplest geometric figures are _______ because the point is ___ and to indicate it we need . Line is _____and therefore to draw it, you need __. The ray is , therefore, in order to build it, we need ___. A segment is ____ and so to build it you need __. Under the plane I understand _____.	*The hypothesis of the study.* I think, the simplest geometric shapes are point, line, plane, because the point is not a signified concept, and to denote it we need touch a page of a notebook with a pen and to identify the point with any letter of the Latin alphabet. The line is not the signified concept, but under the line we understand a geometric figure composed of points, and that's why to draw it, we need to put a ruler on a notebook page and draw a line along the ruler with a pencil. The ray is a part of the line, which is bounded by the point only on one side, and so to build it, we need to mark a point, lay a ruler on a notebook page and draw a straight line from the point along the ruler with a pencil. The segment is a part of the line, which is bounded by the points on both sides, and to build it, we need to mark two points, put a ruler on the notebook page and draw a straight line between the points along the ruler with a pencil. I understand the plane as an unsignified concept, a floor and a ceiling may be the examples of the plane.

Objectives of the study:

1. To clarify the concepts of "point", "segment", "ray", "line", "plane" and define the characteristic features of the construction of each shape.
2. To formulate our own instructions in relation to the construction of geometric figures and using them to construct points, segments, rays, and lines.
3. To provide a method of measuring the length of geometric figures and to perform the task by using it.
4. To draw a conclusion as to which shapes are the simplest geometric figures and create a drawing using an online picture editor.

In the process of systematizing tasks for educational research we use components of learning and research task.

The task for restoring the contents of the text "The historical origin of the term or concept". The task is based on a logical technique, which involves the identification, selection and explanation of the historical origin of the term or the concept.

Performing this type of task, we recommend students to use the dictionary "Brief information on the history of mathematical terms and concepts" or historical references, which are presented in the textbook.

For example. Find the information about the meaning of the word "scale" and "coordinates". Complete the sentences using the words (table 5).

Table 5. A sample of the job

The word "scale" in ___ language and means . Therefore I can conclude that to build the scale I need to find out ____. The word "coordinates" in ___ language means . Therefore I can conclude that the scale must be indicated by _____ and the scale should contain ___.	The word "scale" in the Latin language means "ladder". Therefore I can conclude that to build the scale I need to find out the value of division of the scale. The word "coordinates" is formed by combining two Latin words, which in Latin means "streamlined", "specific" and "singing", "together". Origo – "the beginning" is indicated by the point O. Therefore I can conclude that the scale must be indicated by the beginning and the scale should contain an ordered numerical series.

The task "Explore the cells in your notebook" is aimed at the search, selection, and establishment of relevant cell properties and further for their use during the execution of tasks that promotes the active mental activity of students.

Example. Be observant, setting the degree measures of angles. Using only a ruler build angles of the given degree measures. Formulate conclusions.

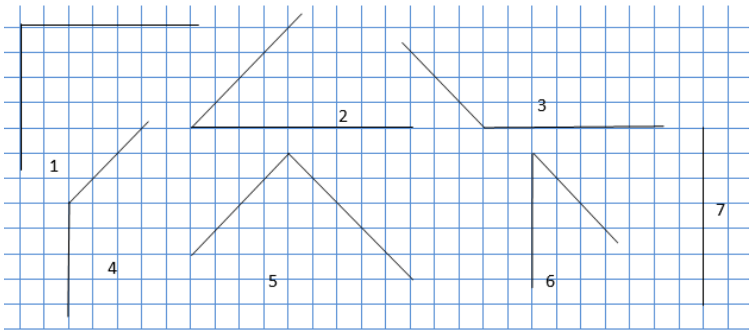


Figure 6. Drawing to the task "Define the degrees of angle measures"

Note to the task. $\angle 1 = 90°$; $\angle 2 = 45°$; $\angle 3 = 135°$; $\angle 4 = 135°$; $\angle 5 = 90°$; $\angle 6 = 45°$; $\angle 7 = 180°$.

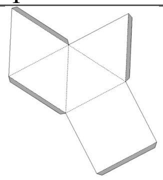
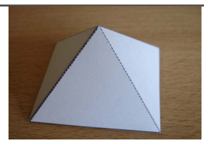
The conclusion to the task. To build an angle, the degree measure of which is equal to 90°, we need: the sides of the angle to coincide with two adjacent sides of the square (cell), or the sides come out of one point of the square (cell) and coincide with the diagonals of the adjacent squares (cells). To build an angle, the degree measure of which is equal to 45°, we need: the sides of the angle should come out from the same point of the square (cell) in the way that one side of the angle coincides with the side of the square (cell) and the other coincides with a diagonal. To build an angle, the degree measure of which is equal to 135°, we need: the sides of the angle to came out from the same point of the square (cell) in the way that one side of the angle coincides with the side of the square (cell) and the other coincides with a diagonal adjacent square (cell). To build an angle, the degree measure of which is equal to 180°, we need to build two rays on the same line with a common beginning.

The objective of the task "Explore a white sheet" is the study of geometric concepts by constructing and studying their models. To perform this kind of task students need to prepare drawing tools, a sheet of white paper, scissors, pencils, and some glue.

Example. Consider, what actions you have to perform with a sheet of paper to make a model of a rectangular parallelepiped, cube, and pyramid.

At the lesson be ready to tell your classmates how to make a model of the geometric figure and to show your version of performing the task (table 6).

Table 6. Recommendations regarding the production model

Task	***To construct a model of a pyramid***
Equipment	A sheet of paper, pencil, scissors, ruler, glue.
Algorithm of performance	1. To take a white sheet of paper. 2. To draw a scan of the pyramid. 3. To cut the scan. 4. To glue the pyramid.
Drawings or pictures of making a model	
 **Figure 7.** Scanning pyramid.	 **Figure 8.** Model pyramid.

The development and generalization of the above objectives has enabled us to organize a series of learning and research tasks for student in adolescence. An example of one of such tasks is presented below as a sample:

The objective of the study. To explore the mutual disposition of straight lines that lie in the same plane, to set their names, to learn how to draw them.

*The hypothesis of the study***.** I think straight lines can ________________________ on the plane.

I know that the lines can be ____________, their form depends on _____________.

To build straight lines it is necessary to use _________________________.

Objectives of the study:

1. Find out the meaning of "parallel lines", "perpendicular lines", and "perpendicular".

2. To formulate students' own instructions regarding the construction of parallel and perpendicular lines and, using them, to draw such lines.

3. To offer a way to recognize the mutual placement of straight lines on a plane and using them to perform the task.

4. To conclude how straight lines can be placed on the plane, to specify the types of lines and create the picture using any graphic editor online.

The procedure of the study

Step 1 / complete on your own. Look carefully at the drawings and sign them. Think about what is similar in the images. Try to draw the subject of the environment, which, in your opinion, can be offered as one more example. Submit your own example.

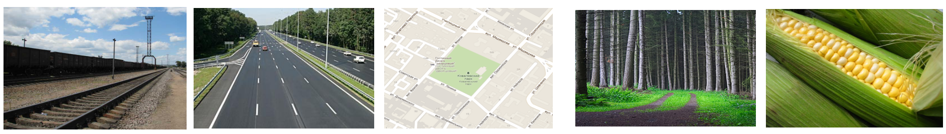


Figure 9. Of environment objects that give an idea of parallel and perpendicular lines

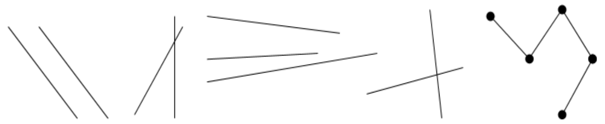
Step 2 / use the dictionary of the origin of mathematical terms and concepts. Revise the meaning of the words "line", "straight line". What information about the line and straight line have you ever heard?

____

Step 3 / do it with a classmate. Find similar facts in the given information about the origin of the terms "perpendicular", "parallel" and the modern interpretations of "direct parallel", "perpendicular lines".

____
_________.

Step 4 / do it orally. Look carefully at the picture. Share your opinion about how straight lines can be located on the plane. Write down the answer and fill in the chart below.



The answer.

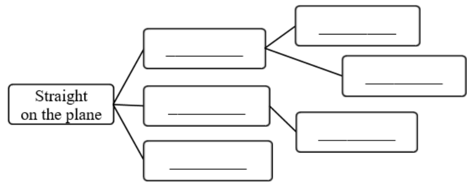


Figure 10. Drawing the task “Step 4”

Step 5 / complete, using the terminology dictionary. Find out the symbolic record of the parallelism and perpendicularity of straight lines. Using it, write down:
The straight line *a* is parallel to the line *b*. _____. The line *a* is perpendicular to the line *c*. ______
The straight line AB is perpendicular to the straight line CK. _____. The straight line OH is parallel to KM _____/

Step 6 / complete it in a pair. Based on the drawings, try to establish a mathematical fact. "Take” information from the drawings. What can you say about the distance and angles between the lines? Write it down in the conclusion.
Conclusion 1.
____.
Conclusion 2.
____.

Step 7 / do it independently. Analyze the content of the previous tasks and look carefully at the drawings. Develop and write down the tips, which, in your opinion, should adhere to construct parallel and perpendicular lines, using the tools below. Write down your ideas completing the following sentences.

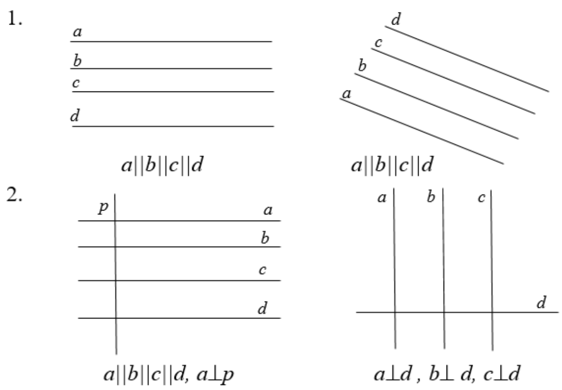


Figure 11. Drawing the task “Step 6”

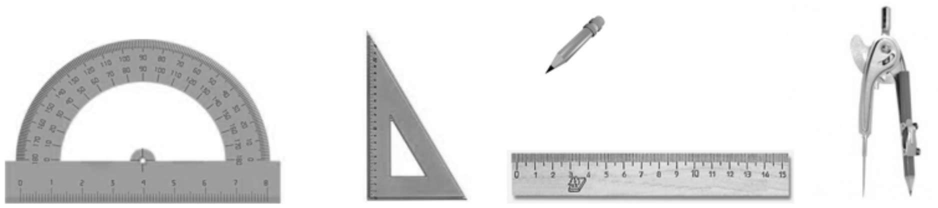


Figure 12. Measuring tools

Tips for constructing parallel lines using ____.

Tips for constructing perpendicular lines using ____________________________________.

Step 8 / / do it independently. Use formulated advice and carry out tasks.

1. Draw two parallel lines.
2. Draw two perpendicular lines.
3. Draw three parallel lines and a perpendicular line to one of the given lines.
4. Draw two lines that cross each other and parallel lines to each of them.
5. Draw two straight lines, that cross each other, and perpendicular lines to each of them.
6. Draw perpendicular lines and parallel line to one of them.

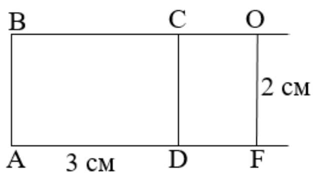
Mark the straight lines drawn by you. Formulate a conclusion and write it down using symbols.

Step 9 / do it independently. Think and draw parallel and perpendicular lines, without using the drawing tools. Mark the straight lines drawn by you. Write the answer using symbols.

Give advice which, in your opinion, would concern the construction of parallel and perpendicular lines in the notebook in cells.

Step 10 / perform orally. Think, solve the problem and formulate the answer. Draw an arbitrary straight line AB i and build the parallel to it SK. Draw a perpendicular line OK to the straight line SK. Will the line OK be perpendicular to line AB?

Step 11 / do it independently. Find the square and perimeter of a quadrilateral ABCD, if you know that AF is parallel to BO. AD equals 3 cm, OF equals 2 cm. Write a brief condition to the task.



Given:
Find:
Answer.

Figure 13. Drawing the task "Step 11"

Step 12 / find a partner to perform this task. Look carefully at each of the drawings. Specify the mutual placement of the lines on the plane. Offer a way to validate your assumptions.

Identify the tips for finding ways of placing straight lines on a plane.

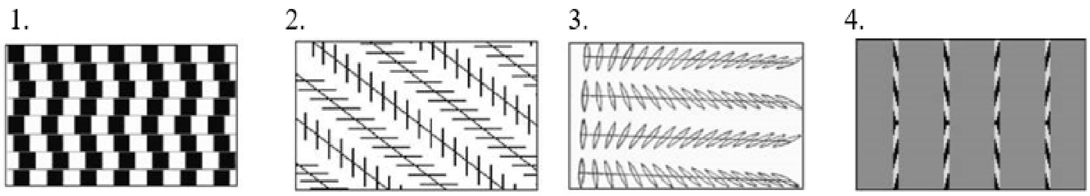


Figure 14. Material design to the task "Step 12"

Tips for finding the mutual placement of straight lines on a plane:

1.______________________________.

2.______________________________.

N.______________________________.

Step 13 / do it with a classmate. Exchange the notebook with a classmate and build three points that are not lying on one straight line. Open your notebook. Look closely at the constructed points. Build two perpendicular lines through these points, use a different color for parallel lines, and use the third color to build perpendicular to a straight line through one of the points.

Step 14 / do it at home. Think about what actions you have to perform with a sheet of paper without using any tools, to demonstrate parallel and perpendicular lines.

At the lesson be ready to tell your classmates how to fold a piece of paper and to show your version of the task.

Step 15 / complete it at home. Dream up and create a drawing that would consist only of straight lines. Use any online graphics editor.

Step 16 / complete it at home. Learn the world of professions (the task from the artist). During wallpapering, we need to build horizontal and vertical straight lines. Determine a valuable fact by answering the question: "How can you to draw horizontal and vertical lines on the wall?"

Step 17 / complete it at home. Analyze and generalize information of the given learning and research task (table 7).

Conclusions. We believe that educational and research activity is student’s activity focused by a teacher as a result of which the students has got formed and generalized methods of solving individually or socially significant problems. As a conclusion: any activity is carried out by solving problems, including learning activity is carried out by solving educational problems, which make up educational and research tasks in a particular system, whose solution is not an end, but is a method to achieve educational purpose.

Work on the implementation of teaching and research tasks, requires the teacher’s coordinating actions. In the first stage of the implementation of the tasks a teacher:

1) performs overall planning of students’ activity and behavior;

2) encourages them to perform specific actions (highlights the purpose and object of research, stimulates the interest);

3) ascertains the degree of understanding of the problem;

4) explains the work requirements, methods, tools, and principles to achieve certain results;

5) predicts performance.

Student’s actions and behavior are diverse: understanding of the task and the desire to fulfill it; questions to the teacher about the content of the task; request to repeat explanations of ways, means and principles of achieving the goal; interpretation of the meaning and content of the task; assessment of complexity, the ability to complete the task; refusal to perform tasks.

During the second phase of the assignment, the relations in the classroom are changed. A teacher’s activity is based on: clarifying or further explanation; providing practical help and support efforts (actions) of the student; stimulating cognitive activity, independence (questions, comments, advice); evaluation of the quality of work; an explanation of errors. The following items are typical for student’s activity: questions, treatment, self-esteem of actions and words, clarifying, debate with the teacher.

Thus, the impact of training depends on the situation of communication during common activities, in the interaction when the task requires concerted actions from the student. The structure of mutual relations between a student and a teacher depends on:

Table 7. Generalization and systematization of competences topics "Perpendicular and parallel lines"

The subject of the study. Perpendicular and parallel lines

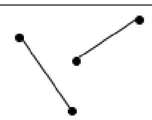
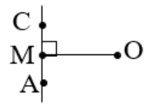
Remember, what mutual placement of the lines on the plane may be. Fill in the table.

	Verbal description	Graphic presentation	Symbolic record
1.			
2.			
N.			

Remember and write down the definition of parallel and perpendicular lines.

____

Read the text in each cell of the table carefully. Fill in the table.

	Verbal description	Graphic presentation	Symbolic record
1.	The line AB is perpendicular to the line KP		
2.			CD⊥MN
3.	The line OH is parallel to FO		
4.	The perpendicular to the line OB is held Through the point M		
5.			
6.			

Think and fill in the gaps in the sentences listed below and give a graphic image.

Verbal description	Graphic presentation
If two lines have the same point, then we say that they ___________	
Two lines intersect in _______________ point	
Parallel lines do not have _______________ points	
If two lines have more than two mutual points, then these lines ___	
Two lines are parallel to the third, ___________________ each other (a sign of parallel lines)	
Through a point that is not lying on a straight line, you can draw ___________________ parallel line	

Reflection.

Questions for discussion with the teacher and classmates	Clarifying hypotheses of the study

choice of teaching methods; the content of educational task; stage of its implementation; efficiency and quality of the work results; evaluation, self-assessment of the task.

It is important for the students to feel a teacher's interest in their common learning and cognitive activity at the lesson which further allows students to develop their creative thinking, ways of learning and builds a personal active position.

4.4. Competence Problems in the Mathematics Course of Basic School and Methods of Their Creation

I. Bogatyreva, O. Kolomiets, O. Terekh, V. Tereshchenko

Among basic educational subjects aimed at building up the multifaceted personality mathematics is one of the most essential ones. In modern society mathematical knowledge is not only thought of as an end in itself but is considered to be a means of a personality development. Therefore, one of the main tasks of the Ukrainian reform in school mathematical education is to strengthen its developmental component.

According to State standard of the second generation and the program of the mathematics for 5-9 classes of general educational institutions (with the changes confirmed by the Ministry of Education and Science of Ukraine in May 2015), the competent approach is laid a basis for the construction of the maintenance and the organization of the process of teaching mathematics at school. The basic idea of this approach is that the main result of education – should be not specific knowledge, skills and abilities, but the ability and willingness of the person for effective and productive activity in various socially important situations. From the standpoint of basic competency approach the subject competence becomes the direct result of the formation of educational activity. The subject competence can be defined as the students' ability to act independently in the situation of uncertainty while solving the pressing problems arising in the course of studying this school discipline.

In this paper we consider the development of mathematical competence among the secondary school pupils. According to S.A. Rakov[1] (2005), mathematical competence is "the ability of person to see and apply mathematics in real life, to understand the content and method of mathematical modeling, to build a mathematical model to investigate methods of mathematics, to interpret the results." The researcher believes that the mathematical competence is defined as levels of educational achievement, which is essential for acquiring mathematical skills, namely the ability of mathematical thinking; the ability of raising and solving mathematical problems; ability to handle mathematical structures; the ability to use mathematical instruments. Generally, the problem of mathematical competence of students is devoted to the work of teachers, trainers, so L.I. Zaitseva[2] (2005), V.A. Starchenko[3] (2009) and others studied the features of formation of mathematical competence in preschool children, S.O. Skvortsova[4] (2006), E.O. Lodatko[5] (2011) and others – elementary school students, N.A. Tarasenkova & V.K. Kirman[6] (2008), O.S. Chashechnykova[7] (1997) and others – secondary school pupils, I.V. Lovyanova[8] (2005), I.Y. Safonova[9] (2014) and others – high school students.

The following mathematical competence items are defined[1] (2005):

---

[1]Rakov, S. A. (2005). *Mathematical education: competence approach using ICT. Monograph.* Kharkiv: Fakt. (in Ukr.).

[2]Zaitseva, L. I. (2005). *Developing elementary mathematical competence in preschool children. Dissertation.* Kiev. (in Ukr.).

[3]Starchenko, V. A. (2009). *Developing logical-mathematical competence in older preschoolers.* Kiev: Svitych. (in Ukr.).

[4]Skvortsova, S. O. (2006). *Methodical system of teaching primary school pupils how to solve story problems. . Monograph. Odesa: Astroprynt. (in Ukr.).*

[5]Lodatko, E. O. In S. T. Zolotukhina (Ed.). (2011). *Mathematical culture of the elementary school teacher. Monograph.* Rivne. (in Ukr.).

[6]Tarasenkova, N.A. & Kirman, V.K. (2008). *The content and structure of mathematical competence of the secondary school students. Mathematics in school*, 6, 3-9. (in Ukr.).

[7]Chashechnykova, O. S. (1997). *The development of mathematical abilities of the secondary school pupils. Dissertation.* Kiev. (in Ukr.).

[8]Lovyanova, I. V. (2005). *The content of mathematics education and the problem of formation of intellectual skills of high school students*. Cherkasy: Publishing department CNU named after Bogdan Khmelnitsky. (in Ukr.).

[9]Safonova, I. Y. (2014). Mathematical competence as a factor in preparing the high school students for life. *The current problems of public administration, education and psychology*, *2*, 65-68. (in Ukr.).

• procedural (the development of mathematical abilities, possession of the skills of mathematical modeling provides the ability to describe (using mathematical models) natural phenomena and processes, finding optimal solutions to controversial professional tasks, etc.);

• logical (the development of mathematical thinking, possession of the deductive research method allow the specialist to analyze and logically build strategies, successfully solve non-standard professional tasks and situations, etc.);

• technological (requires knowledge of the essence of innovative technologies and formation of mathematical skills to use them to solve professional tasks);

• research (the development of creative mathematical thinking and research skills mastery enable the specialist to apply mathematical tools in the study of biological, environmental, physical, chemical, and other professional tasks effectively);

• methodological (methodology ensures mastery of mathematical research).

Mathematical competence involves two components: substantive (mathematical knowledge) and procedural (ability to solve problems). A competent student does not only understand the essence of the objectives offered, but also knows how to solve them, that is – to have a method of "knowledge plus skills". Thus, the mathematical competence involves the following skills: the ability to apply knowledge and skills to solve unfamiliar problems; the ability to solve the unfamiliar problems,; the ability to generalize the results.

Clearly, the attention of researchers is attracted by the contents of the competencies and their assimilation. In order to complete this, it is necessary to introduce such methods, organizational forms and educational means for encouraging the personality development of the pupils, in particular, their mathematical thinking. Since mathematical thinking gets its most intensive progress during the process of solving various mathematical problems, we face an important issue: building a system of problems of a certain orientation and teaching the pupils how to solve them. According to modern requirements they are competency tasks.

A competence task is defined as a kind of educational material conducive to the formation of students' competencies at three levels: substantive, interdisciplinary and main[10] (Dybova & Maslova, 2011). The purpose of the application of such problems in the learning process is:

• creation of a system of universal educational activities;

• providing conditions for the application of knowledge and skills in new, unfamiliar situations for interdisciplinary students;

• acquisition of vital experience among solutions of life problems.

In teaching mathematics, namely, through a mathematical problem "How to determine the center of the circle?" a competency task must be "Desktop backgrounds form a circle. Furniture-makers are to make a hole in the center of the tops to tie a leg. How to determine the center of the desktop using only a ruler with divisions and angles? How to determine the center of the desktop using only a compass? How to determine the center of the table without any tools, if its model is cut out of paper? ". Noteworthy is that the structure of this problem is different.

The structure of the competence problem involves two components[11] (Tarasenkova, Bogatyreva, Kolomiets & Serdyik, 2015): 1) the condition of the problem in the form of a certain unfamiliar life situation; 2) the system of questions that should be answered for

[10]Dybova, M. V. & Maslova, S. V. (2011). Aim and pithy aspect of the concept of "Competence task". *Bulletin of the Volga University named after V. N. Tatishcheva, 8*. (in Rus.).

[11]Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M. & Serdyik, Z. O. (2015). *The structure and content of teaching kits in algebra for the 7th Form.* Science and education a new dimension III (26), 64, 12-18. Budapest: SCASPEE. (in Ukr.).

these terms. For example, in the collections of tasks to test subject competencies[12] [13] [14] [15] (Tarasenkova, Bogatyreva, Kolomiets & Serdyik, in Tarasenkova (Ed.), 2015; Tarasenkova, Bogatyreva, Kolomiets & Serdyik, in Tarasenkova (Ed.), 2015; Tarasenkova, Globin, Bogatyreva, Kolomiets & Serdyik, in Tarasenkova (Ed.), 2015; Tarasenkova, Burda, Bogatyreva, Kolomiets & Serdyik, in Tarasenkova (Ed.), 2015) we used stories: about the after-school lessons at a music school, a swimming pool, of other clubs or sports clubs, and the time spent in such employment; of tariff plans of the telephone company and the choice of favorable tariffs depending on the time of the day and the duration of the calls; on the calculation of the cost of travel in a taxi, rental of equipment; on the range, the time and cost of travel; purchase of a variety of new things, products, furniture, depending on available funds; to place candy in a box of some form, dishes on a table or furniture in the room; on the content of nutrients in foods and the calculation of the daily value of their consumption; situations in the classroom, where two students decided to do the same problem (not necessarily in different ways), and evaluating the correctness of their decision, and others. Also there were offered the plots on the basis of the fabulous, fantastic or imaginary situations. There were no problems of the type "make similar or the offered plan." Like in real-life situations, students should be smart, intelligent, and show other general cultural qualities. Here is an example of competence problems.

Problem 1. Apples are very useful. They contain various number of micronutrients, vitamins and a small amount of ascorbic acid, organic acids, a lot of pectin and fiber. An apple of an average weight of 150g contains about 15 grams of natural sugar.

1) How much natural sugar does a kilo of apples contain on average?

2) How many apples contain on average 300 grams of natural sugar? The European scientists believe that average use of sugar for a healthy adult is about 60 g.

3) How many apples should be eaten during the day to recharge a daily rate of sugar?

4) The content of natural sugar in bananas is about 12% of their weight. How many bananas weighing 250 g are necessary to eat during the day to recharge daily rate of sugar?

5) Is it better to eat 2 apples, or 2 bananas or an apple and a banana to replenish the daily allowance? Explain your answer. Vitamin C, which is contained in products, increases energy, improves health and immune system. The daily rate of consumption of this vitamin is about 50 mg.

6) What is the least number of medium apples, that are necessary to eat in order to supplement the daily requirement of vitamin C, if 100 grams of apples contain 10 mg of vitamin C?

During the process of solving, the problem can be referred to as a developmental one, provided the pupils, while solving them, learn to compare familiar and unfamiliar facts, combine and think, generalize the obtained solutions, and draw conclusions. These are the problems, whose way of solution is not evident, and in order to find it one has to apply heuristics. The main purpose of employing the competency tasks is enhancement of mathematical thinking. In the process of solving these problems, students do not only acquire knowledge and skills of a purely objective nature, but also

[12]Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M. & Serdyik, Z. O. In N. A. Tarasenkova (Ed.). (2015). *Verification of subject competencies. Mathematics, Grade 5.* Collection of tasks to the evaluation of educational achievements of students. Kiev: Orion. (in Ukr.).

[13]Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M. & Serdyik, Z. O. In N. A. Tarasenkova (Ed.). (2015). *Verification of subject competencies. Mathematics, Grade 6.* Collection of tasks to the evaluation of educational achievements of students. Kiev: Orion. (in Ukr.).

[14]Tarasenkova, N. A., Globin, O. I., Bogatyreva, I. M., Kolomiets, O. M. & Serdyik, Z. O. In N. A. Tarasenkova (Ed.). (2015). *Verification of subject competencies. Algebra, Grade 7.* Collection of tasks to the evaluation of educational achievements of students. Kiev: Orion. (in Ukr.).

[15]Tarasenkova, N. A., Burda, M. I., Bogatyreva, I. M., Kolomiets, O. M. & Serdyik, Z. O. In N. A. Tarasenkova (Ed.). (2015). *Verification of subject competencies. Geometry, Grade 7.* Collection of tasks to the evaluation of educational achievements of students. Kiev: Orion. (in Ukr.).

the experience of their practical application. It should be noted that the objectives include information from other subject areas (biology, geography, physics, and chemistry). Quantity, sequence and place of competency tasks in the system of the lesson aims are established by a teacher according to the goals pursued during a certain lesson. Along with this it is necessary for each topic of the mathematical course to contain not less than one third of developmental problems of their total quantity.

Till recently the majority of school problems have been training ones, aimed at forming conscious and profound skills of applying mathematical knowledge. In particular, in the 5th and 6th grades the calculation problems prevailed, for calculation skills have been considered the basic ones. The Ukrainian school education reforms, observed these days, provide some positive changes. A new program in mathematics has been adopted; new course books have appeared. When analyzing the problems in these course books one can conclude, that the quantity of competency tasks has increased. However, the ratio between them and the tasks of other types, to our mind, remains inadequate.

Thus, we face a necessity to supply the existing set, suggested by a course book, with additional competency tasks. These tasks can be fulfilled by a teacher in three ways: selecting the competency tasks from supplementary materials; composing the problems by themselves, taking into account the specificity of a given class; creating the competence task, and intensifying the developmental function of the problems from the current course book.

For the first direction we offer the teacher to use collections of tasks to test subject competences[16] [17] [18] [19] (Tarasenkova, Bogatyreva, Kolomiets & Serdyik, in Tarasenkova (Ed.), 2015; Tarasenkova, Bogatyreva, Kolomiets & Serdyik, in Tarasenkova (Ed.), 2015; Tarasenkova, Globin, Bogatyreva, Kolomiets & Serdyik, in Tarasenkova (Ed.), 2015; Tarasenkova, Burda, Bogatyreva, Kolomiets & Serdyik, in Tarasenkova (Ed.), 2015).

For the second direction a teacher's competency objectives may be based on their own experience and additional literature. However, this work requires considerable effort and time. Therefore, this option is suitable for the teachers who have been working for over 10 years.

For the third direction, we suggest the techniques that allow enhancing the developmental function of the problems taken from the course books in mathematics.

Techniques for enhancing the developmental function of mathematical problems. To them we refer[20] (Bogatyreva, 2008):

- construction of different statements of a problem;
- broadening the set of questions on the problem situation;
- solving the problem in different ways;
- substitution of numerical values by letter symbols;
- composing a problem.

Employment of the techniques mentioned in the work with the problems allows us, firstly, to encourage the purposeful development of mathematical thinking (namely,

---

[16] Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M. & Serdyik, Z. O. In N. A. Tarasenkova (Ed.). (2015). *Verification of subject competencies. Mathematics, Grade 5.* Collection of tasks to the evaluation of educational achievements of students. Kiev: Orion. (in Ukr.).

[17] Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M. & Serdyik, Z. O. In N. A. Tarasenkova (Ed.). (2015). *Verification of subject competencies. Mathematics, Grade 6.* Collection of tasks to the evaluation of educational achievements of students. Kiev: Orion. (in Ukr.).

[18] Tarasenkova, N. A., Globin, O. I., Bogatyreva, I. M., Kolomiets, O. M. & Serdyik, Z. O. In N. A. Tarasenkova (Ed.). (2015). *Verification of subject competencies. Algebra, Grade 7.* Collection of tasks to the evaluation of educational achievements of students. Kiev: Orion. (in Ukr.).

[19] Tarasenkova, N. A., Burda, M. I., Bogatyreva, I. M., Kolomiets, O. M. & Serdyik, Z. O. In N. A. Tarasenkova (Ed.). (2015). *Verification of subject competencies. Geometry, Grade 7.* Collection of tasks to the evaluation of educational achievements of students. Kiev: Orion. (in Ukr.).

[20] Bogatyreva, I. M. (2008). About enhancing the developmental function of tasks in mathematical course in the 5th and 6th grades. *Mathematics in school*, 6, 27-32. (in Ukr.).

such its constituents as analysis, generalization, abstracting, planning, and reflexion); and secondly, to raise pupils' activity during the lesson. Let's study some peculiarities of the above techniques arising during their application to problem solving in studying the main school mathematics.

1. Construction of different statements of a problem. This technique implies the creation of various forms of the written statement of a given problem (Figure 1-4): figure, schematic notes, graphical scheme, flow-graph, table, diagram, etc. Such procedure stimulates the interest to the problem, trains to conduct a profound analysis of the situation, allows pupils to clearly present it, makes the presupposed result pictorial, and conveys the information in a concise form.

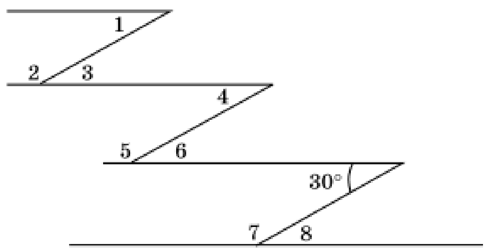


Figure 1.

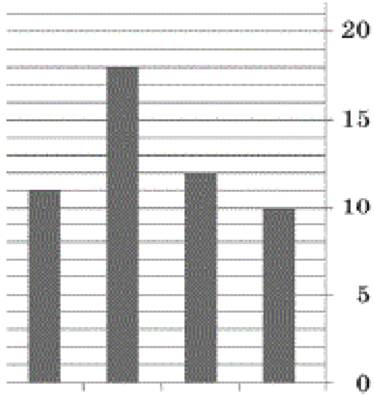


Figure 2.

Figure 3.

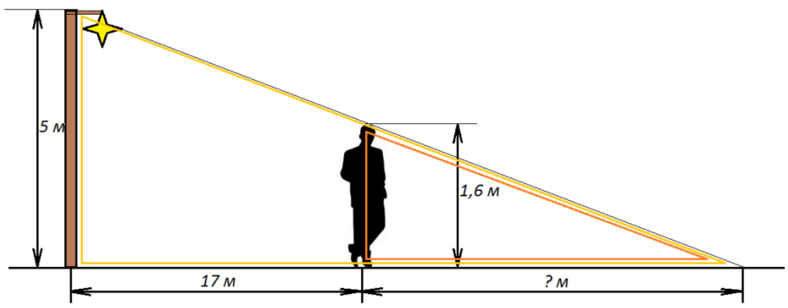


Figure 4.

2. Broadening the set of questions on the problem situation. This means constructing various questions for the problem situation. The questions can be of two types: those requiring answer during the process of solution, and those, the answers to which originate some new problems. It should be noted, that the diversity of the questions is quite important here, e.g.: "How many…", "Calculate…", "Compare…", "Find the regularity…", "Is it possible that…", "If we change… then…" and others.

According to N.A. Tarasenkova[21] (2002), the ability to formulate questions in many cases serves not only for expanding a certain way of solution but also for the choice of the way itself. The variety of questions raises the interest to the problem, as pupils start to better understand its content and see the dependences between all values.

3. Solving the problem in different ways. Applying this technique, a teacher offers the pupils to solve the problem in different ways, choose the most efficient solutions, and substantiate their choice. It can be widely used, while solving text problems in arithmetic and algebraic ways, and when carrying out calculations.

4. Substitution of numerical values by letter symbols. The pupils are offered to read a statement of a problem, substitute all numerical values by letters, and solve the problem with the help of ordinary reasoning. Note, that this is the technique that develops the ability of abstracting and generalization most effectively.

5. Composing a problem. In the application of this technique there are two possibilities: a teacher sets up mathematical problems, based on those given in a course book, or asks the pupils to do this. Furthermore, the problems can be similar to the given one in the statement, formulation of requirements, or inverse to it. It is advisable that the problems stick to different content: calculation, movement, cooperation, geometrical and others. In solution of the obtained problems one should emphasize those in which the way of solution is analogical to the original problem.

In the performed investigation we have found that the above indicated techniques can be used both separately and in the combined manner. This depends on the goal of the lesson, the level of the class in general and the level of each individual pupil.

Consider an example of how a combination of techniques can help to enhance the developmental function of the training problem taken from a school course book[22] (Tarasenkova, Bogatyreva, Bochko, Kolomiets & Serdyik, 2013), made it competent.

Problem 2. *Two cars simultaneously started moving towards each other from two points. The distance between the villages is 260 km. They met in 2 hours. Define the speed of each car, if the speed of one of them is 10 km / h is more than the speed of the other.*

After analyzing the problem situation, technique 1 implies that the pupils are able to write down a short statement using a table or a graphical scheme. After that a teacher offers to solve the problem in different ways (technique 3): arithmetic and algebraic ones.

[21]Tarasenkova, N. A. (2002). *Using sign and symbolic means in teaching mathematics. Monograph.* Cherkassy: "Vidlunnya-Plus". (in Ukr.).

[22]Tarasenkova, N. A., Bogatyreva, I. M., Bochko, O. P., Kolomiets, O. M. & Serdyik, Z. O. (2013). *Mathematics, Grade 5. The textbook for secondary schools.* Kiev: VD "Osvita". (in Ukr.).

On the next stage it is appropriate to formulate additional questions (technique 2).

– Which of the cars was traveling at higher speed? How much higher?

– Which of the cars has traveled a longer distance to the place of meeting? How much longer? Explain, why.

– What was the distance between the cars an hour after they started moving?

– What will the distance between the cars be three hours after they start moving, if they will move in one and the same direction? Explain the received answer.

– Compare the distances between the cars for one and three hours after they start moving. Explain the answer received.

To apply technique 5 one can offer the pupils to consider the movement of cars in opposite directions, or in the same direction.

In our opinion such problems will acquire cognitive character, if you enter additional information – the name of items from which cars have left, briefly describe one or another of their attraction.After this work in class, solutions of competetive problems do not cause difficulties for students. For example, the next task from the collection.

Problem 3. Tarasova gora or the Chernecha mountain is the mountain in the town of Kaniv, a part of the Shevchenko National Reserve "Tarasova Gora", the burial place of the Ukrainian poet Taras Shevchenko.

In summer a group of tourists went on a trip to the mountains. First they traveled by bus from Poltava to Cherkasy, making a stop in Kremenchuk. Then they spent the night in a tent camp near Cherkasy. The next day they sailed on a boat on the Dnieper to Kaniv. The next day they returned to Poltava from Cherkasy.

1) Before the stop in Kremenchuk the tourists have been going by bus for 1 hour and 30 minutes, which is 45 km less than when they continued their travel after the stop and were on the way for 2 hours. Determine the speed of the bus at each section of the path, knowing that after the stop the speed was 5 km per hour higher.

2) A group of 24 people settled in single and double tents. There were 16 broken tents. How many of them were single ones and how many double ones?

3) Kaniv is at a distance of 65 km up the Dnieper from Cherkasy. Find a speed of the boat when going to Kaniv and back, knowing that the travel to the Tarasova Gora takes 2 hours and 30 minutes, and the travel back takes takes 6 minutes more.

4) Can you find out what is the speed of the Dnieper River in the Kaniv district? Explain the answer.

5) How long will it take you to get from Cherkasy to Kaniv by bus at a speed of 70 km/h if the distance by the highway is 5 km longer than that by the Dnieper?

A highly essential component of the work at a problem is the reflexion of educational activities performed by the pupils. Questions like “What do you think?”, “Why do you think so?” train the pupils to substantiate all their assumptions and draw conclusions while reasoning.

It is worth noting, that the use of the combined techniques increases the educational and developmental function of the problem to a great extent. Search for different ways of the solution, formulation of the questions on the problem situation, and the solution of the composed problems makes the pupils the conscious participants of the educational process, develops the components of mathematical thinking.

It has been determined, that the techniques shown can be applied to many problems of the mathematical course of the main school.

CHAPTER FIVE

THE MATHEMATICAL PREPARATION IN TERTIARY EDUCATION

5.1. Application of the portal ORTUS in Studying Mathematics at the Riga Technical University

I. Volodko & S. Cernajeva

Introduction. Nowadays with the intensification of the use of Technologies, the demand for qualified specialists who can be competent in the newest scientific achievements, modern technologies, and materials, can create qualitative global business. Therefore, the education system nowadays should be ready to offer young people such education that they could be competitive in these new working conditions in order to increase their level of competences.

Review of the latest publications. Nowadays computers and other information Technologies are more and more interfering into our everyday life and the field of education is not an exception. Information and communication technologies are more and more used in the fields of higher education and training[1] (Crampton, Vanniasinkam & Milic, 2010). The mathematics learning environment is also influenced by the development of information Technologies[2] (Galbraith & Haines, 1998). Students and graduates welcome the use of ICT in high schools[3] (Breen, Lindsay, Jenkins & Smith, 2001). The literature[4] (Dunlosky, Rawson, Marsh, Nathan & Willingham, 2013) describes in detail 10 main teaching methods and evaluates their relative usefulness. Student testing is mentioned as one of the main methods, its usefulness has gained the highest approval.

The main mathematics teaching problems at the Riga Technical University (RTU) and their solutions are described in this article[5] (Cerrajeva & Volodko, 2012). The main problem is an insufficient level of student knowledge in Elementary Mathematics. The article[6] (Volodko & Cerrajeva, 2015) describes the situation in RTU and dwells on some activities used by the teaching staff of the Department of Engineering mathematics aimed at solving these problems.

The article[7] (Volodko & Roscina, 2012) describes the portal ORTUS and the possibilities it provides to the students and at the lecturers of mathematics.

[1]Crampton, A., Vanniasinkam, T., Milic, N. (2010). Vodcasts! How to unsuccessfully implement a new online tool. *Interaction in Communication Technologies and Virtual Learning Environments: Human Factors*, 118-128.

[2]Galbraith, P., Haines, C. (1998). Disentangling the nexus: Attitudes to mathematics and technology in a computer learning environment. *Educational Studies in Mathematics*, 36 (3), 275-290.

[3]Breen, R., Lindsay, R., Jenkins, A., Smith, P. (2001). The Role of Information and Communication Technologies in a University Learning Environment. *Studies in Higher Education*, 26 (1), 95-114.

[4]Dunlosky, J., Rawson, K.A., Marsh, E.J., Nathan, M.J., Willingham, D.T. (2013). Improving students' learning with effective learning techniques: Promising directions from cognitive and educational psychology. *Psychological Science in the Public Interest, Supplement*, 14 (1), 4-58.

[5]Cerrajeva, S., Volodko, I. (2012). Teaching Mathematics in Riga Technical University. *International Conference "Mathematical Competence Development in Higher Education Institutions"*, Latvia, Jelgava, 14 December, 2012, 21.

[6]Volodko, I., Cerrajeva, S. (2015). Improving Knowledge of Elementary Mathematics – the Way to Better Studying of Higher Mathematics. *Rural Environment. Education. Personality (REEP): Proceedings of the 8th International Scientific Conference*, Latvia, Jelgava, 15-16 May, 2015. Jelgava: The Real Estate Market Development Impact on Life Quality - Main Aspects, Tendencies and Important Regulations, 401-406.

[7]Volodko, I., Roscina, I. (2012). The Use of ORTUS Portal for Teaching Mathematics in Riga Technical University. *Some topical problems of the modern mathematics and mathematical education: scientific conference materials „Gercenovskis readings – 2012”,* Russia, SanktPeterburg, 16-21 April, 2012. SanktPeterburg: *the Library of the Academy of Science*, 168-173. [In Russian]

The application of mathematical software in Latvia is described in the article[8] (Roscina & Volodko, 2014). In the articles [9] [10] (Volodko & Dzenite, 2009; Volodko & Roscina, 2014) it is highlighted in more detail how the lecturers of the Department of Engineering mathematics apply the package of program MATHEMATICA in teaching mathematics.

In the article[11] (Volodko & Cerņajeva, 2016) we try to address essential issues arising from the use of the tests: Do students treat doing the test tasks in earnest? Do tests provide really objective results and what should be done to obtain them? The study method used at the paper is the analysis of the mathematics test results. The statistics of results is obtained from the tests performed by the first year students of the Faculty of Computer Science and Information Technology of the Riga Technical University.

The article[12] (Cerņajeva & Volodko, 2016) focuses on it how the test materials were worked out and tried on the open online site platform *mooc.rtu.lv*, which gives students a chance to improve their knowledge by studying mathematics in a new, simple, and effective way. It helps students and pupils understand basic mathematical notions and get prepared for mathematics tests, it also gives support for learning mathematics unaided.

The articles[13] [14] (Kislenko, Grevholm & Lepik, 2007; Philippou & Christou, 1998) deal with the problem of improving the methodology of teaching mathematics and making the process more interesting and effective.

Methodology, results and discussion. Being focused on the quality of higher educational institutions in Latvia and their improvement, the Riga Technical University, starting from the academic year of 2007/2008, has transferred to the united e-learning system of RTU in the portal ORTUS, which is based on the e-learning program MOODLE (Modular Object-Oriented Dynamic Learning Environment). ORTUS in Latin means ”new beginning”. The RTU portal ORTUS (*www.ortus.lv*) facilitates the life of every student of RTU as long as they want to perform their learning more effectively, more progressively and quickly. With ORTUS one can reach their university from any place and at any time. ORTUS is a portal where RTU students can:

- get information about studies and see the records of their academic assessments;
- read the lecture summaries and other study materials;
- have access to scientific databases;
- read messages and news;
- use the RTU forum as a way of communication, communicate with other RTU students;

[8]Roscina, I., Volodko, I. (2014). Using of Computer Algebra Systems in Latvia and Riga Technical University . *Papers of international scientific conference „Education, science and economics at higher educational establishments and schools. Integration into international educational environment”*, Armenia, Cahkadzor, 24-29 March, 2014. Cahkadzor, 511-514. [In Russian]

[9]Volodko, I., Dzenite, I. (2009). The Use of the Package "Mathematica" in Teaching of Mathematics in Riga Technical University. *Proceedings of the 2009 International Conference on Engineering and Mathematics*, Spain, Bilbao, 17-19 June, 2009. Bilbao: Javier Bilbao Landatxe, 75-79.

[10]Volodko, I., Roscina, I. (2014). Using the Package MATHEMATICA for Teaching Mathematics in Riga Technical University . *Papers of international scientific conference „Education, science and economics at higher educational establishments and schools. Integration into international educational environment”,* Armenia, Cahkadzor, 24-29 March, 2014. Cahkadzor, 482-485. [In Russian].

[11]Volodko, I., Cerrajeva, S. (2016). Assessment of Students' Knowledge by Means of Tests. *Rural Environment. Education. Personality. (REEP): Proceedings of the 9th International Scientific Conference*, Latvia, Jelgava, 13-14 May, 2016. Jelgava: Latvia University of Agriculture, 328-333.

[12]Cerrajeva, S., Volodko, I. (2016). Improvement of Teaching Methodology of Mathematics for Students and Pupils Using the MOOC Platform. *15th International Scientific Conference "Engineering for Rural Development": Proceedings. Vol.15*, Latvia, Jelgava, 25-27 May, 2016. Jelgava, 1286-1290.

[13]Kislenko, K, Grevholm, B., & Lepik, M. (2007). Mathematics is important but boring: students’ beliefs and attitudes towards mathematics. In C. Bergsten, B. Måsøval , & F. Rønning (Eds.), *Relating practice and research in mathematics education. Proceedings of Norma05, Fourth Nordic Conference on Mathematics Education, Trondheim, 2nd-6th of September 2005*. Trondheim: Tapir Akademisk Forlag. 349–360.

[14]Philippou, N. G., & Christou, C. (1998). The Effects of a Preparatory Mathematics Program in Changing Prospective Teachers’ Attitudes towards Mathematics. Educational Studies in Mathematics, 35, 189-206.

- get acquainted with RTU laws and regulations;
- create one's own e-mail.

Each subject has a separate environment where it is possible to cooperate both with group mates and the teaching staff, and to find out about the topics covered in the subject as well as the requirements for a successful course pass.

In the portal ORTUS the following courses prepared by the Department of Engineering mathematics are available in the e-learning section: Mathematics I, Mathematics II (semesters 1 and 2), Discreet mathematics, Supplementary mathematics. Beginning from the academic year of 2008/09 all lecturers of the department work with the ORTUS system. The RTU ORTUS e-learning environment ensures convenient access to electronic study materials and electronic tests.

Changes in the practical studies of engineers are necessary because of complicated technical and organisational work combinations joined together by the work of an engineer; moreover, engineering projects even more tightly will be supported by social and political considerations and will be connected with new business models. Therefore, it would be necessary to implement several important criteria in the changes of practical studies:

- sustainability – within the projects and work of engineers it requires systematic and integrated approach in order to decrease social and environmental influence of engineering;
- complexity – the increase of volumes of information, complexity of projects and systems may infer some unpredictable consequences;
- cooperation at work – results in all new business forms should be improved by learning to cooperate and not to compete;
- risk management – to be aware that complicated projects can create unpredictable consequences in dimensions of health, environment, safety etc., therefore, critical thinking and knowledge in these fields are essential;
- ability to see opportunities – the complicated present projects require talented engineers who could find effective solutions to problems and to satisfy a client's needs.

For the teaching staff of the Faculty of Civil Engineering and the Faculty of Computer Sciences and Information Technology, the question about application of acquired mathematical knowledge in the professional activities of specialists becomes very topical. The answers to the following questions are being looked for: what we are to teach and how to do it so that the acquired mathematical knowledge could be applied in professional activities. The level of student training, their intellectual abilities and cognitive advantages should be considered while organising practical studies. The proportion of contact hours and individual work should be reckoned with in accordance to the level of difficulty and importance of the topic in other mathematics-related academic disciplines.

The University mathematics teachers should develop a training system based on new technologies which could allow students find application for their mathematical knowledge even during the process of studies.

One of the ways to implement new Technologies into the educational process is to teach students mathematical software and its application. Since the 80s of the previous century the integrated system for the automatization of mathematical calculations MATHCAD has gained broad popularity. It was developed by the company Mathcad Soft (USA). Within the system the mathematical calculations are performed using general mathematical formulas and symbols.

The popularity of the program is confirmed by the fact that for the last years several versions of this program have appeared in the market. The program works in the Windows environment, is quite simple in use as the form of mathematical expressions correspond to the general style.

The system MATHCAD similarly to other software is being continuously developed and improved thus obtaining wide choice of versions. By MATHCAD one can get results for both simple and complicated tasks, including:

- to perform arithmetical operations using build-in functions and mathematical operators;
- to define variables and functions;
- to evaluate the change of values of functions and expressions in the range of argument change;
- to quickly construct function charts of one or two arguments ;
- to perform operations with matrixes;
- to perform differentiation of functions;
- to calculate sums and integrals;
- to solve equations and equation systems expressed numerically.

The tasks can be solved both numerically and analytically using the method of specific symbols. Data and results can be shown graphically.

The students of the Faculty of Civil Engineering of RTU are shown the work with MATHCAD during the lectures. However, a computer cannot solve a task which is not conceived by the students themselves when they are not able to formulate the algorithm of the solution. Even though such algorithm is formulated, it should be entered in the computer in the form comprehensible for the latter.

Pedagogues-constructivists let students test new ideas, evaluate information, and find new solutions in different situations themselves. The context of studies is also essential; it should facilitate creative thinking and critical approach[15] (Geidzs & Berliners, 1999).

During the period from 2006 through 2014, the students of the Faculty of Computer Sciences and Information Technology of RTU were taught the package of programs MATHEMATICA besides traditional training of mathematics. For the last two years we have started working with the package of programs MATLAB. For two semesters the first year students have laboratory works with computers where students learn to solve tasks with the help of mathematical software similar to those solved in practical studies. During the lab classes the students learn with the help of MATHEMATICA or MATLAB:

- to assign functions and calculate their values with the necessary precision;
- to perform operations with matrixes and vectors;
- to draw a line in plane given in Descartes coordinates, polar coordinates or parametrically;
- to draw a line or surface in space;
- to solve algebraic equations or equation systems;
- to calculate the limit of a function and use to determine its asymptotes;
- to find derivative and apply it in the research of functions;
- to calculate definite, indefinite or multiple integrals, as well as to apply it in practical task solutions;
- to solve differential equations;
- to expand a function into power series and to find the domain of convergence of functional series.

The acquired knowledge of MATLAB is improved during the second year within the course of Numerical methods.

As experience shows, students master mathematical software with interest. The work with it helps students to understand the algorithm of the task easier. Many students with low success in mathematics and who are not able to solve the task on paper unaided,

[15]Geidzs, N. L., Berliners, D. C. (1999). Pedagogical psychology. *Riga; Zvaigzne ABC*. 1-662. [In Latvian].

manage to solve these tasks quite well in the environments of MATLAB, MATHCAD or MATHEMATICA. Moreover, using mathematical software students can solve time-consuming tasks at short notice, whereas the solution on paper would take much time and effort.

The use of the ICT capacities in mathematics class facilitates the work of the teachers, as well as the study process is made more interesting and effective. Thus, it also improves the quality of higher education and the level of readiness for the labour market.

The assessment of students' knowledge through online tests is also a relatively new method of knowledge assessment and control. It brings wide modernization and optimization opportunities into the teaching and learning process.

As always, new ideas, tendencies and methods bring about contradictory judgements. Although the application of tests can be only relatively considered as a new method, still quite opposite characteristics of tests, their usefulness, and objectivity of evaluation can be heard. Sometimes the same arguments can be heard from both sides, e.g. the ones who are pro and the ones who are against testing.

Arguments "against":

- tests reduce the role of the lecturer;
- tests do not allow to evaluate of knowledge in detail and in depth, etc.;

Arguments "pro" :

- tests reduce the labor-intensity of the academic process;
- tests allow to objectively evaluate students' knowledge, etc.

The question arises, "Does testing really give objective results and what should be done to get them?"

Testing itself usually does not create any substantial problems. The important component parts of this process are:

1) determination of aims under test;
2) correct choice of question types;
3) precise formulation of questions;
4) evaluation and interpretation of testing results.

The aims should be set before the test is developed, identifying what we want to achieve with that test[16] (Appleby, Samuels & Treasure-Jones, 1997). If there are several aims, the level of importance should be defined; less important aims should be eliminated, but for each important aim an equal number of questions should be prepared.

Choosing a type of a question, it should be considered that for multiple choice questions (these are the questions where students can choose one answer among the given versions) the compiler should foresee students' mistakes. Therefore, it is better to choose questions requiring students' own answers. A question should be short and clearly formulated, simple, so that students can answer it by using their course book, by remembering facts and algorithms[17] (Boesen, Lithner & Palm, 2010). It is even better if a question can be compiled so that it also verifies interim results and traces the mistakes made[18] (Seemann, 2015).

The evaluation and interpretation of the results is also an essential component of testing, because without overall evaluation of the results it cannot be concluded where to pay attention further, how to improve methods and the quality of study process. Mathematical and statistical methods are widely used to evaluate the results of tests which allow to use computers at this stage and thus automate and optimize the performance of due tasks.

---

[16] Appleby, J., Samuels, P., Treasure-Jones, T. (1997). Diagnosis – A knowledge-based diagnostic test of basic mathematical skills. *Computers and Education*, 28 (2), 113-131.

[17] Boesen, J., Lithner, J., Palm, T. (2010). The relation between types of assessment tasks and the mathematical reasoning students use. *Educational Studies in Mathematics*, 75 (1), 89-105.

[18] Seemann, E. (2015). Unit testing Maths automated assessment of mathematic exercises. *Lecture Notes in Computer Science*, Volume 9307, 530-534.

Within the last years the lecturers of the Department of Engineering mathematics compiled and realised several tests in the environment of ORTUS used as a replacement of almost all homework in the first semester. During the semester of the previous academic year the first year students were to do 14 tests in total. All tests are not labour-consuming; they consist of 2-5 tasks. In all the tasks of all the tests students are to write the correct answer, and not to choose among versions of answers. Each test is to be done in 2 hours, after that the test will be automatically closed. Each test allows three trials; the final evaluation is the best score of all trials.

The analysis of the tests made by students showed:

- the greatest part of students (approximately 80%) performed those tests;
- if a student has not gained the maximum points for the first trial, they try again;
- on average, 81% of those who performed the test got a maximum score;
- on average, 62% of students who performed the test got maximum at the first trial;
- the majority of students prefer performing tests on ORTUS to doing the homework in pen.

To create the effective study process, the lecturer should be aware of the level of knowledge of their students in the respective subject. As experience shows, many students enrolled to the Riga Technical University do not have sufficient basic mathematical knowledge to study higher mathematics and other exact and technical subjects successfully. During the first University mathematics class the elementary mathematical knowledge was tested; it was established that almost half of all students are not able to solve two of five simple tasks. The test consists of five simple tasks: operations with fractions, calculation of the value of functions, expression of a variable in a linear connection, basic properties of powers and logarithms. Each correct task gives 2 grades. The test is passed if a student scores at least 4 grades.

Figure 1 shows the proportional division of grades for students who started their studies at RTU in this academic year 2015/16. The data summarised the results of the tests in elementary mathematics with 1387 students performing the tests.

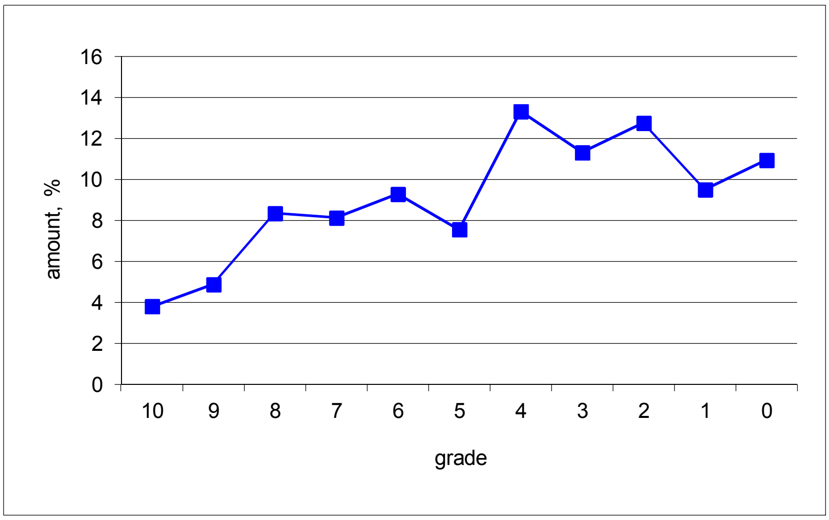


Figure 1. Division of grades for the test of the knowledge of elementary mathematics in the 2015/16 academic year

Comparing the results of the last 8 years (Figure 2), we see that during the last two years the results have improved. The results are not good enough, all students enrolled in the technical high school should pass such test successfully.

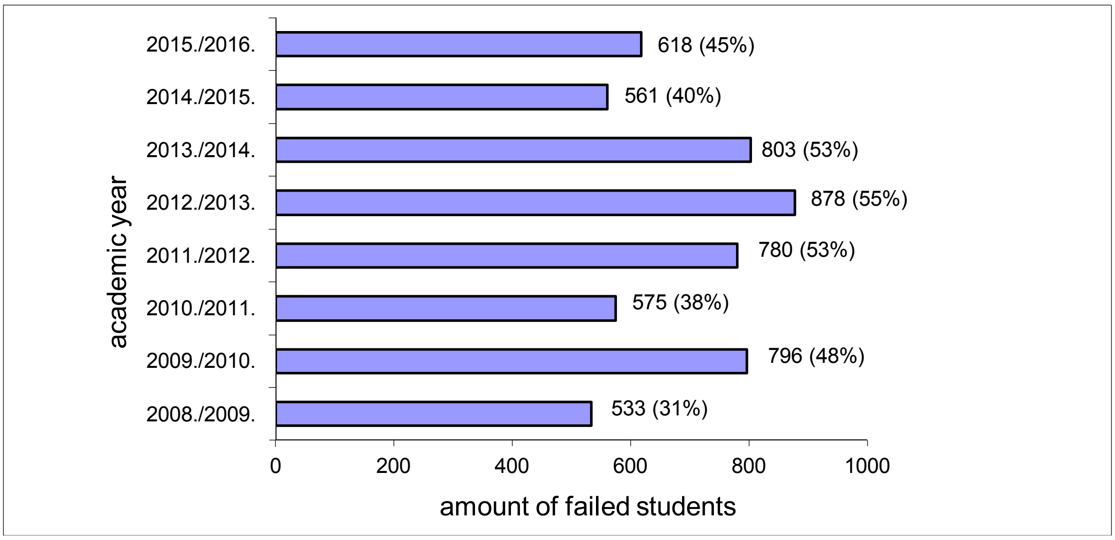


Figure 2. The amount of the students who failed the test in elementary mathematics during the last eight years

To help students with poor knowledge of elementary mathematics, the RTU performs several different activities:

• a course of video lectures on elementary mathematics is created and is freely accessible on YOUTUBE;

• an intensive course on elementary mathematics was organized before the academic year started;

• a new subject "Basic sections of elementary mathematics" is developed ;

starting from the next academic year there will be planned obligatory additional lectures;

• a supplementary course on elementary mathematics was created on the online platform *mooc.rtu.lv*.

Uniting several structural units of RTU in the autumn of 2015, a new open online platform *mooc.rtu.lv* was created. At present, a supplementary course on elementary mathematics is available on the platform and gives the opportunity to the students to improve their knowledge by learning mathematics in a new, simple, and interactive way. There are six different topics of the study program of elementary mathematics within that supplementary course. Each topic is divided into three parts – theory, video materials, and tests. To pass the course successfully, a user should pass all available tests (the lowest evaluation possible four grades). The test questions can be answered only once, hence – students should be attentive and considerate. The course can be accessed using the ORTUS password. It is planned that soon also the supplementary courses of higher mathematics and physics will be available on the platform.

The course involves the sections of the elementary mathematics that are needed to master higher mathematics successfully. The aim of the course is to ensure interactive, topical, available additional educational material for students and pupils.

Topics of the adjuvant course in basic mathematics are:

1. Arithmetic operations (addition, subtraction, multiplication, division) with fractions.
2. Algebraic expressions, their transformations. Simplification of algebraic expressions. Formulas of abbreviated multiplication.
3. Linear and quadratic equations, linear and quadratic inequalities.
4. Interval of function definition and function value. Simple function values. Even and odd functions. Ways of assignation of function. Graphs of basic functions.
5. Basic properties of powers.

6. Exponential equations and inequalities. Logarithmic equations and inequalities. Basic properties of logarithms, their usage for solving equations and inequalities.

7. Trigonometric functions. Signs of trigonometric functions. Basic trigonometric identities.

The main problem for students is created by the amount of repeatable material, therefore, both the application of visual aids and favourable atmosphere for communication are to be successfully organised, it would raise cognitive desire in students and their willingness to gain positive results. To ease the process of reviewing, great attention is paid to the application of visual aids. Online courses can be a wonderful facilitator of the professional development where everybody can choose appropriate pace for themselves. In the MOOC course the study material is visualised at most, the material can be seen both in general and in detail. The basics of theory and explanations are given alongside with the examples of the task solving patterns. The material is a tool for a student who wants to get short and particular information, whose attitude towards obligatory homework is negative, but they are willing to receive study materials which would be appropriate for their independent work – concise, comprehensive, with prepared tasks. The summary of the statistics shows that free access massive open online course method or MOOC has been becoming more and more popular worldwide; however the question of improving the students' results is still open, as the universities of the world which offer such courses conclude that only one tenth of students pass these courses successfully.

The higher school has already understood that a student comes from school and their interest in exact sciences should be developed already there, in a secondary school. An alarming fact is that by the data of the Ministry of Education and Science for 2015 almost 60% of students chose to study social sciences and humanities, while only 21% – engineering and natural sciences. Such tendency is related to the fact that secondary school graduates consider social sciences as the easiest solution. However, personnel advisors emphasise that in the future the greatest demand in the labour market will be for the specialists in exact sciences. Therefore, using the Internet appreciated by pupils, RTU wishes to open broader opportunities to master exact sciences effectively by the portal users.

On the 8th of January 2010 Riga Technical University presented the new portal ORTUS to the secondary school students and teachers of. Its aim is to create a group of interested persons and to let pupils duly try the role of a University student in order to help choose the most appropriate study program. The portal will support everyone willing to study regardless of the place of residence – this will be the opportunity to find out about student's life in RTU without visiting the higher school.

There is a mathematics course created for pupils in the RTU ORTUS portal which offers to acquire mathematics independently and to prepare oneself for the centralised examination (CE) in mathematics.

On the portal high school students can master the following themes for the preparation course of mathematics:

1. Numbers and operations with them.
2. Algebraic expressions, their identical alterations.
3. Concept of the function. Linear function, quadratic function, exponential function.
4. Algebraic equations. Equation systems with two variables.
5. Algebraic inequalities. Systems of inequalities.
6. Power function, its properties. Exponential equations, exponential inequalities.
7. Logarithmic function, its properties. Logarithmic equations. Logarithmic inequalities.
8. Trigonometric functions, its properties. Trigonometric equations. Trigonometric inequalities.

9. Elements of combinatorics. Concept of probability.
10. Vectors in plane, operations with them.
11. Planimetry.
12. Stereometry.

Genuine interest to the materials of the mathematics section of the portal is proved by the number of participants of the preparation course for the CE in mathematics – 12232 (as of 01.09.2010).

The portal supports everyone willing to study regardless of their place of residence – it is an opportunity to find out about the life of a student of RTU without going to the capital city. With the ORTUS portal we wish to create a dialogue with pupils, to help start studies at the university and thus to make higher engineering technical education more available for every young person.

It is important to awake interest and prepare a young person for University academics when the students are still in high school, so that they are sure of their readiness and correctness of their choice. E-learning portal ORTUS is a great support for a modern secondary school student in mastering exact sciences deeper and creating links with the life at RTU.

The portal ORTUS is an integral part of the quality management system of studies of RTU, ORTUS now is the most modern, large-scale and multifunctional University portal in Latvia. Every week it is used by more than 10000 users. It should be mentioned that the Service of Information Technologies of the Riga Technical University has obtained the acknowledgment certificate of the prise "Platinum mice 2009" about excellent achievements in the development of the portal ORTUS of the RTU, issued by Latvia Association of Information and Communication technologies.

Conclusions.

1. As motivation is the aggregation of reasons that evokes and maintains active learning process of the mathematics studies, assistance to the students in structuring and planning the learning process is necessary for fostering motivation.
2. The ORTUS portal is an important part of the quality management system of the RTU academics and helps both students and teaching staff in the academic process.
3. Use of the computer mathematical systems brings innovation into the teaching of mathematics and makes it more attractive. The work with MATLAB, MATHCAD or MATHEMATICA helps students to understand mathematics easier.
4. The assessment of students' knowledge by means of testing enables wide modernization and optimization opportunities of the educational process, but it cannot be the only evaluation method, though properly compiled tests and proper evaluation system of the results in combination with other assessment methods provide objective results.
5. Students prefer performing the tests rather than handing-in the homework in writing to the lecturer. More than half of the students, who perform the tests repeatedly, are trained to solve the problems and are gaining highest ratings as the result. Both the students and the faculty treat employing the tests in the academic process positively.
6. The authorial designed and tested material in an open online site platform *mooc.rtu.lv* can help students and pupils understand separate mathematical questions, give support for acquiring mathematics independently, help prepare for mathematics tests and as a result promote pupils' motivation for an active acquisition of mathematics.
7. Pupils' learning motivation can be increased by using the designed material. This work is a great gain for a lecturer, because the preparation stage of a lecture is a very time-consuming process.
8. Critically viewed, the MOOC courses are not revolutionary ones. Yes, world-class universities offer courses for free to any student that is by now the only significant difference from traditional full time or distance education. But pedagogical methods that

are mainly used in these courses are not revolutionary – video lectures, tests with limited answers.

9. Pedagogical researches show that the most effective learning methods are those where students have many opportunities to get into discussions with a lecturer.

10. It is important to awake interest and prepare a young person for studies the latter is still a secondary school student. The ORTUS portal is a great support for a modern secondary school student to master exact sciences deeper and to create links with the life at RTU.

11. Expedience and the necessary amount of the subjects that are included in the University curriculum are evaluated by the input that each subject renders to achieve the common study goals of the program. To ensure a good quality study process, knowledge, skills and competences that will be provided by acquiring the corresponding subject, as well as their usefulness for the employer's assessment, must be acknowledged. The quality of the students' competences is not dependent only on the amount of knowledge in some subject, but also on the acquired ability and skill to solve problems independently. That is why the goal of teaching mathematics is firstly determined by the necessity for the students to be prepared to practically use the mastered subject in life, and that is why the academic process should be purposefully planned.

5.2. Case-study of Mathematics Diagnostic Testing of the Ukrainian Engineering Students

G. Lutsenko

Introduction. The key aspect of the research in engineering education is the recognition of the special role of mathematics in the training of future engineers. However, a specific place of mathematics can cause an extra challenge and complexity of the research activity. The research into mathematical training of the engineering students goes under the influence of a number of facts. Thus, mathematics is a compulsory subject for most universities; it is among the entrance requirements for engineering undergraduates; furthermore, mathematics and statistics are the fundamental components of university training. Numerous engineering subjects have a highly mathematical content, and students have different problems in its acquisition as well as with the use of mathematical skills in engineering context. Such a challenge was recognized by many universities in Europe, and now researchers from many countries are actively engaged in researching innovative approaches to enhance the learning experience of students.

The quantitative analysis of current trends in acquiring mathematical skills by the students who enter the third level programmes has been reported in[1] [2] (Faulkner, Hannigan & Gill, 2010; Gill, O'Donoghue, Faulkner & Hannigan, 2010). Besides, the engineers' vision of necessary characteristics of mathematical competence[3] [4] [5] (Kent & Noss, 2003; Goold & Devitt, 2012; Alpers, 2013) as well as the effective ways of curriculum reorganization in engineering education are the important issues concerning existing demands.

An integrated approach to the analysis of mathematical training and the development of relevant curriculum is presented in the report of the Mathematics Working Group (MWG) of the European Society for Engineering Education (SEFI)[6] (Alpers, 2013). The study of MWG is based on the concept of mathematical competence[7] (Niss, 2003) and contains the list of content-related learning outcomes. The description of competence-oriented assignments in the mathematics education of engineering students and some examples are presented in[8] (Aplers, 2014).

Therefore, a set of factors affecting the mathematical training of undergraduates can be divided into two categories. General factors are important for most universities and

---

[1]Faulkner, F., Hannigan, A., Gill, O. (2010). Trends in the mathematical competency of university entrants in Ireland by leaving certificate mathematics grade. Teaching Mathematics and its Application. 29(2), 76-93. doi:10.1093/teamat/hrq002

[2]Gill, O., O'Donoghue, J., Faulkner, F., Hannigan, A. (2010). Trends in performance of science and technology students (1997–2008) in Ireland. International Journal of Mathematical Education in Science and Technology. 41(3). 323-339. doi: 10.1080/00207390903477426.

[3]Kent, P., Noss, R. (2003). Mathematics in the University Education of Engineers. A Report to The Ove Arup Foundation. Retrieved from http://www.lkl.ac.uk/research/REMIT/Kent-Noss-report-Engineering-Maths.pdf.

[4]Goold, E., Devitt, F. (2012). The role of mathematics in engineering practice and in the formation of engineers. In Avdelas A. (Ed.) Proceedings of SEFI 40th annual conference "Engineering Education 2020: Meet the Future". Thessaloniki, Greece.

[5]Willcox, K., Bounova, G. (2004). Mathematics in engineering: Identifying, enhancing and linking the implicit mathematics curriculum. Paper presented at 2004 Annual Conference of American Society for Engineering Education. Annual Conference&Exposition. Salt Lake City, USA.

[6]Alpers, B. (2013). A Framework for Mathematics Curricula in Engineering Education. Retrieved from http://www.sefi.be/wp-content/uploads/Competency%20based%20curriculum%20incl%20ads.pdf.

[7]Niss, M. (2003). Mathematical competencies and the learning of mathematics: The Danish KOM project. In Gagatsis A, Papastravidis S, (Eds.) Proceedings of the 3rd Mediterranean Conference on Mathematics Education. Athens, Greece.

[8]Aplers, B. (2014). Competence-oriented assignments in the mathematical education of mechanocal engineers. Proceedings of SEFI 42nd annual conference. Birmingham, UK. Retrieved from http://www.sefi.be/conference-2014/0012.pdf.

specific factors depend on the requirements of specified degree programmes. So, the level of mathematical competence of high school graduates changes in the learning environment; the permanent increase of demands to engineering profession put forward by scientific and technological advance can be referred to the general affecting factors. Specific factors include diverse needs for mathematical competencies in the case of different engineering profile, the level of interaction between engineering and mathematics faculty and the readiness of academic staff to introduce modern pedagogical methods.

The issues listed above are also topical for the Ukrainian system of higher education. Despite the fact that Ukraine joined the Bologna Process in 2005, the implementation of reforms has not been systematic. Nowadays, special attention of the Ukrainian higher educational institutions (HEIs) is paid to the need to develop the degree programme curricula considering the competence-based approach as well as state-of-art teaching, learning and assessment methods. The use of the accepted techniques for developing a curriculum should be combined with the analysis of the actual state of students' skills.

This study intends to investigate the possibilities of mathematics diagnostic testing implementation in the system of the Ukrainian engineering education. This paper also presents an analysis linking the results of External Independent Testing, Grade Point Average and academic achievements of engineering students at Bohdan Khmelnytsky National University of Cherkasy. The ways of mathematics curriculum improvement are considered.

Background. From the mentioned above, it is inferred that the development of mathematics curricula in the engineering education is directly related to the level of mathematical competence of high school graduates. The study results[9] (Culter & Pulko, 2002) show that it is mathematics that is identified as the most problematic subject for the first- and second-year students. The engineering subjects with high mathematical content are also found to be rather difficult. The implementation of diagnostic tests is an effective tool that is often used at a number of universities. The report[10] (LTSN MathsTEAM Project, 2003) contains the data of surveys considering the UK departments using diagnostic testing. Diagnostic tests to identify the lack of mathematical skills and the obtained results are helpful for both students and academic staff. The detailed statistical analysis of trends in mathematical competency based on the data of diagnostic testing in mathematics conducted at Ireland universities during the whole decade is presented in[11] [12] (Faulkner, Hannigan & Gill, 2010; Gill, O'Donoghue, Faulkner & Hannigan, 2010).

The methodology and results of mathematics diagnostic test carried out at Dublin Institute of Technology is described in[13] (Carr, Bowe & Ni Fhloinn, 2013). Although the Diagnostic Test is based on the syllabus of the first-year mathematics modules, testing is conducted for the students of different grades. In order to decrease the guessing of correct answer, the technique of 'paired' questions is used. In 2013, a project aimed at the comparison of mathematical preparedness of the first-year engineering

---

[9]Culter, G.L., Pulko, S.H. (2002). Investigation UK undergraduate electrical and electronic engineering attrition. International Journal of Electrical Engineering Education. 39(3). 181-191.

[10]LTSN MathsTEAM Project. (2003). Diagnostic Testing for Mathematics. Retrieved from http://www.sigma-network.ac.uk/wp-content/uploads/2013/12/diagnostic_test.pdf.

[11]Faulkner, F., Hannigan, A., Gill, O. (2010). Trends in the mathematical competency of university entrants in Ireland by leaving certificate mathematics grade. Teaching Mathematics and its Application. 29(2), 76-93. doi:10.1093/teamat/hrq002

[12]Gill, O., O'Donoghue, J., Faulkner, F., Hannigan, A. (2010). Trends in performance of science and technology students (1997–2008) in Ireland. International Journal of Mathematical Education in Science and Technology. 41(3). 323-339. doi: 10.1080/00207390903477426.

[13]Carr, M., Bowe, B., Ni Fhloinn, E. (2013). Core skill assessment to improve mathematical competency. European Journal of Engineering Education. 38(6). 608-619. doi: 10.1080/03043797.2012.755500

students in Ireland and Portugal was initiated[14] (Carr, Fidalgo & Bigotte de Almeida, 2015). Such project gives an opportunity to identify both mutual and particular trends in mathematical preparedness of engineering students. Looking for the balance between content-related and behaviour-related competences during tests question development is an important issue, particularly, in case of engineering students.

Therefore, a few stages of diagnostic testing implementation can be established: developing test questions according to current requirements, choosing testing procedure and handling results with the obligatory support of feedback. The outcome processing of the test results should be used, primarily, to organize some support to the students 'at risk', and to improve learning process in general. Hereby, feedback is provided and testing can be used as an element of curriculum development.

The system of secondary education in Ukraine is in a state of transformation. There are three levels of secondary education in Ukraine: Primary School (4th grade), Basic Secondary School (9th grade) and Senior Secondary School (11th grade)[15] (NORRIC Country Report, 2009). The final state examination at the end of senior secondary school is conducted at schools and includes three subjects: the Ukrainian language and literature (compulsory subject), History of Ukraine or Mathematics (chosen by the student) and one more optional subject. Supplement to the Certificate of Complete General Secondary Education contains students' grade points which are used to calculate Grade Point Average (GPA).

In 2008 the Ukrainian Ministry of Education introduced a new assessment model for secondary school leavers and called it the External Independent Testing (EIT)[16] (USETI Final Report, 2009). EIT replaced the locally organized oral and written entrance examinations to higher education institutions. Nowadays, admission to HEIs is conducted based on the rules of admission. The rules contain the list of the competitive subjects in the EIT certificate and the minimal scores required for the admission to training programs. Mathematics is a compulsory subject for all engineering undergraduate degree programs.

In 2008-2014, the norm-based reporting scale for EIT was used with the grades of 100-200. The minimum main subject level required for admission was 140 grades, and for the non-core subjects it was 124 grades. The rules of admission to a particular institution allow of establishing higher scores, but most HEIs picked out minimum level. The competitive score of the applicants was calculated by adding the points of ETI certificate and GPA calculated on 100-200 scale.

In 2015, some changes in mechanism of the EIT organization and assessment were introduced. Firstly, the modified technique of EIT results estimation was used. Scale from 100 to 200 points was also used, but, only those students whose results were higher than some "threshold value" were included into a rating list. Therefore, in 2015 the minimum passing score could be 100 points. Besides, two levels of complexity for tests in the Ukrainian Language and Mathematics were introduced: Basic and Advanced. Mathematics tests on Basic and Advanced levels consisted of 30 questions (48 points) or 36 questions (66 points), respectively. The main goal of introducing two levels of complexity was to take into account the difference in the actually required level for future specialization of graduates.

---

[14] Carr, M., Fidalgo, C., Bigotte de Almeida, M.E. (2015). Mathematics diagnostic testing in engineering: an international comparison between Ireland and Portugal. European Journal of Engineering Education. 40(5). 546-556. doi: 10.1080/03043797.2014.967182

[15] A Nordic Recognition Network (NORRIC) Country Report. (2009). The Educational System of Ukraine. Retrieved from http://norric.org/files/education-systems/Ukraine2009.

[16] The Ukrainian Standardized External Testing Initiative (USETI) Final Report. April 2007 – December 2009. Retrieved from http://pdf.usaid.gov/pdf_docs/Pdacq648.pdf.

In 2015, the competitive score of an undergraduate was calculated by adding the points of EIT certificate and GPA taken with ratio quotient whose sum equalled 1. The consequences of implementing such innovation were mixed. So, just few top universities required the EIT certificate with the Advanced level of Mathematics test. According to the statistics of the Ukrainian Centre for Educational Quality Assessment[17] (UCEQA, 2015), about 12% of graduates chose the advanced level of mathematics test. Particularly, such situation is caused by the lowering of the interest to difficult sciences among secondary school leavers which is observed during the last decade. Thus, the universities are very cautious about the rules of admission.

According to[18] (USEQUA, 2015) EIT the Basic Mathematics test covers the following topics: algebra and elements of analysis (numbers and expressions, equations, inequalities and their systems, functions) – 19 questions, elements of combinatorial analysis and elements of the probability theory and statistics – 1 question, geometry (planimetry, stereometry) – 10 questions.

Methodology. Currently the Bohdan Khmelnytsky National University of Cherkasy (ChNU) offers a Bachelor Degree in five engineering specialties. The Institute of Physics, Mathematics and Computer-Information Systems (IPMCIS) offers a Bachelor of Engineering in Automation and Computer Integrated Technologies (ACIT) and in Applied Mathematics (AM). The Faculty of Computing, Intelligence and Control Systems offers a Bachelor of Engineering in Computer Sciences (CS), Software Engineering (SE) and in System Analysis (SA). Thus, about 400 students study towards a Bachelor Degree and about 80 students study towards a Master Degree. The list of competitive subjects in the EIT certificates according to the rules of admission to CNU in 2015[19] (Rules of admission to Cherkasy National University in 2015, 2015) is presented in Table 1.

Table 1. The list of competitive disciplines in EIT certificates according to the rules of admission to CNU in 2015

Speciality	Competitive disciplines	Ratio Quotient	Minimum score
Automation and Computer Integrated Technologies	1. Ukrainian Language 2. Mathematics (Basic level) 3. Physics	0.2 0.4 0.3	101 110 101
Applied Mathematics Computer Sciences Software Engineering System Analysis	1. Ukrainian Language 2. Mathematics (Basic level) 3. English Language	0.2 0.4 0.3	101 110 101

An average score in Mathematics and difference between maximum and minimum scores for the students entering CNU in 2015 are presented in Table 2. There is a considerable irregularity of student results. Range of scores varies from 47 to 61.5 points. The data corresponding to 2013-2014 are presented in Table 3. As it was mentioned above, that system of EIT assessment was changed in 2015. Hence, we used the ratio between scores that were received in EIT and percentage of correct answers according to the EIT Math Syllabus. Such fact complicates the comparative analysis of

[17]Ukrainian Centre for Educational Quality Assessment (UCEQA). (2015). Report on external independent testing of learning outcomes for Senior Secondary School graduates who wish to enter higher education institutions in Ukraine in 2015. Retrieved from http://testportal.gov.ua/reports.

[18]Ukrainian Centre for Educational Quality Assessment (USEQUA). (2015). Syllabus of External Independent Assessment in Mathematics. Retrieved from http://testportal.gov.ua/prepare_math/

[19]Rules of admission to Cherkasy National University in 2015 (CNU). (2015). Retrieved from http://cdu.edu.ua/info/abiturientam/pryimalna-komisiia/2015-07-07-11-58-5.html.

the applicants' preparedness in different years. EIT test consisted of 30 questions in 2015, 34 questions in 2014 and 33 questions in 2013.

Table 2. Average EIT Certificate score in mathematics (standard deviation) and range for students entering engineering programmes in 2015

Speciality	Number of students	Average EIT Certificate score	Standard deviation	Range
Automation and Computer Integrated Technologies	20	175.85	16.33	61.5
Applied Mathematics	16	173.66	14.36	50.5
Computer Sciences	20	176.3	15.61	58
Software Engineering	15	184.13	15.14	49
System Analysis	14	174.14	15.67	47

Table 3. Average EIT Certificate score in mathematics (standard deviation) and range for students entering engineering programmes by year

Speciality	Year	Number of students	Average EIT Certificate score	Standard deviation	Range
ACIT	2013	26	172.64	14.8	56.5
	2014	25	166.54	9.69	45.0
Applied Mathematics	2013	19	165.05	8.94	34.5
	2014	21	172.85	12.38	47.5
Computer Sciences	2013	25	166.1	8.37	37.5
	2014	25	165.52	14.2	45.5
Software Engineering	2013	20	178.78	11.05	38
	2014	20	174.53	9.96	32.5
System Analysis	2013	15	169.5	8.8	27.5
	2014	15	173.1	8.28	32.5

The statistical software package SPSS was used for data analysing. Figure 1 shows the distribution of EIT scores expressed in the percentage of correct answers for engineering students of all five programmes in 2013-2015.

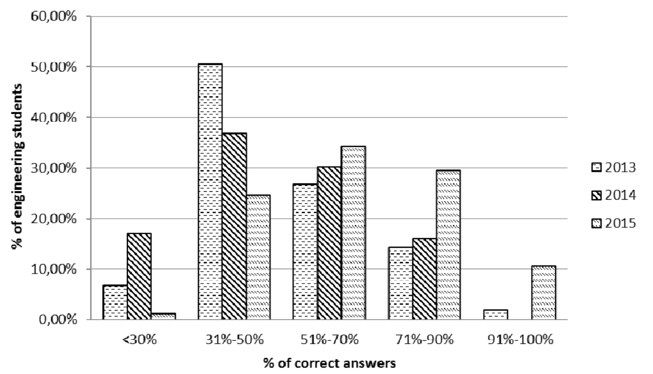


Figure 1. Distribution of EIT scores expressed as a percentage of correct answers.

Besides, the attempt of the correlation analysis was carried out in order to investigate the correlation between EIT Mathematics scores of students and their examination grades in Higher Mathematics for the second-year and third-year students of ACIT. ACIT curriculum includes two examinations in Higher Mathematics at the end of the first and fourth semesters, respectively, and test at the end of the second semester. Table 4 shows the Pearson's correlation coefficients between examinations scores and EIT Mathematics scores and corresponding statistical values. The obtained results of correlation analysis are different. Concerning the second-year students of ACIT, correlation is weak and it isn't statistically significant. As for the third-year students of ACIT, the correlation quotients vary within the range from 0.359 to 0.557.

Such results show the fact that analysis and using of EIT Certificate data should be supplemented with some additional methods of mathematical knowledge and skill assessment.

Table 4. Coefficient of correlation between EIT Mathematical Scores and academic achievements of ACIT students in 2013-2015

	N	Year	Coefficient of correlation and statistical significance (two-tailed)		
			Higher Mathematics (Examination 1)	**Higher Mathematics (Final Test 1)**	**Higher Mathematics (Examination 2)**
ACIT	25	2	0.308 p=0.137	0.306 p=0.134	
ACIT	26	3	0.557 p=0.003	0.359 p=0.072	0.412 p=0.037

At the first stage of the diagnostic testing organization the Syllabus of External Independent Assessment in Mathematics[20] (UCEQA, 2015), curricula of degree programmes in ACIT and AM and the experience presented in[21 22] (Carr, Bowe & Ni Fhloinn, 2013; Carr, Fidalgo & Bigotte de Almeida, 2015) were analysed. Some general core topics being similar for different countries were identified in all the cases analysed. Moreover, we tried to outline both the list of the main topics and skills required for further learning. Consultation process with academic staff working mainly with the first-year students was organized. In discussion, the conception of key mathematical competences was used[23] (Aplers, 2014). It includes, in particular, the ability to think mathematically, to solve pure and applied mathematical problems and to handle mathematical symbols. Such active interaction of engineering, physics and mathematics is considered as an important component of test design.

At the first stage, the trial testing in mathematics was carried out in order to estimate the choice of topics list, complexity of questions, time limits and system of test assessment. The proposed trial test covers the following areas: operation with fractions, system of linear equation, quadratic equation, planimetry, logarithms, indices, differentiation, applying algebra to physical and engineering problems (writing and solving equations), and writing mathematical expressions by programming. The paper-based format of testing with paired questions was decided to use. Trial diagnostic test consists of 30 questions with four answers, one of which is correct. The duration of the test is 40 minutes. The trial test was given to three focus groups including the first- and second-year ACIT students (17 and 24, respectively) and the second-year AM students (9).

The results of the focus groups testing were analysed with the idea of estimating the complexity factor of each question. We consider the complexity factor to be the percentage of incorrect answers. Questions with lower values of complexity factor (closer to zero) were partially replaced or reformulated in order to avoid the option of mere guessing the correct answers. At the same time, in our opinion, complete

[20]Ukrainian Centre for Educational Quality Assessment (UCEQA). (2015). Report on external independent testing of learning outcomes for Senior Secondary School graduates who wish to enter higher education in Ukraine in 2015. Retrieved from http://testportal.gov.ua/reports.

[21]Carr, M., Bowe, B., Ni Fhloinn, E. (2013). Core skill assessment to improve mathematical competency. European Journal of Engineering Education. 38(6). 608-619. doi: 10.1080/03043797.2012.755500

[22]Carr, M., Fidalgo, C., Bigotte de Almeida, M.E. (2015). Mathematics diagnostic testing in engineering: an international comparison between Ireland and Portugal. European Journal of Engineering Education. 40(5). 546-556. doi: 10.1080/03043797.2014.967182

[23]Aplers, B. (2014). Competence-oriented assignments in the mathematical education of mechanocal engineers. Proceedings of SEFI 42nd annual conference. Birmingham, UK. Retrieved http://www.sefi.be/conference-2014/0012.pdf.

exclusion of question with the complexity factor less than 0.2-0.1 is not relevant because such questions could be used as an indicator. The general structure and the duration of diagnostic test carried out on the second stage of testing were not changed. Such diagnostic test was given to the third-year ACIT students (22), the first- and third-year AM students (12 and 12, respectively).

Results and discursion. The results of the focus groups testing are presented in Table 5. The percentage of correct answers for all groups and the distribution of scores are presented.

Table 5. Marks received by focus groups in diagnostic test (stage 1)

Course	Year	Number of students	Average scores (%)	Students <50% (%)	Students 50%-75% (%)	Students 76%-90% (%)	Students >90% (%)
ACIT	1	17	75	17.65	11.76	58.82	11.76
ACIT	2	24	80	4.17	25	54.17	16.67
AM	2	9	82	-	22.22	55.56	22.22

More than a half of the focus group students' scores over 75% in trial test. The obtained results are also interpreted as some indicators of question complexity. As shown on Figure 2, fractions, planimetry and applications to real world problems are among the best answered topics for all the of students. Volumes and areas, logarithms and differentiation are among the most complicated.

The results of the second stage of testing are presented in Table 6.

Table 6. Marks received in diagnostic test (stage 2)

Course	Year	Number of students	Average scores (%)	Students <50% (%)	Students 50%-75% (%)	Students 76%-90% (%)	Students >90% (%)
ACIT	3	22	70	13.63	50	31.82	4.55
AM	1	12	72.2	25	16.67	50	8.33
AM	3	12	71.9	25	16.67	58.33	-

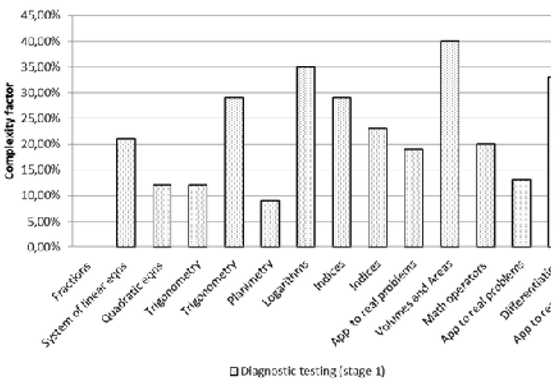


Figure 2. Complexity factors of diagnostic test questions (stage 1).

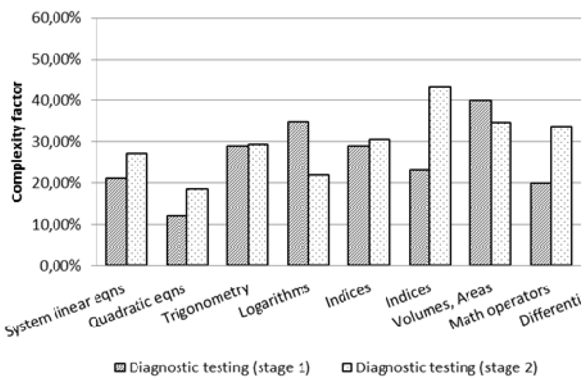


Figure 3. Complexity factors of diagnostic test questions that remain unchanged (stage 2).

Figure 3 shows the complexity factor of questions that remain unchanged and Figure 4 shows the complexity factor of questions that was reformulated on the second stage of testing. Figure 4 shows that there is a significant variation in students' results in differentiation and indices despite the fact that the second group of students includes the third-year ACIT and AM students. At the same time, the second group shows better results in logarithms. Trigonometry is among the worst-answered questions. Figure 5 shows the distribution of diagnostic test scores expressed in the percentage of correct answers by years.

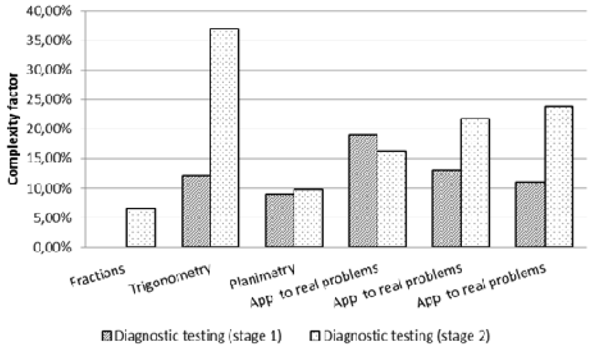


Figure 4. Complexity factors of reformulated test questions (stage 2).

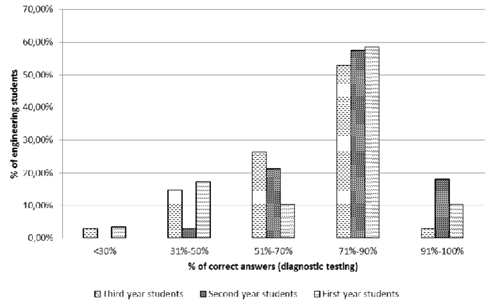


Figure 5. Distribution of diagnostic test scores expressed as a percentage of correct answers by years.

Conclusions and Research Prospects. The development issue of modern engineering curriculum has a whole set of dimensions because the academic faculty have to analyse not only the level of knowledge but also the level of preparedness of our students to future study and professional activity. The analogy with the building of wall, where the content-related competences and behaviour-related competences are interpreted as bricks and as mortar, respectively may be used. Furthermore, in the case of engineering education, the use of the applied mathematical problems that are closely related to further professional activity of the engineering students are widely used.

The conducted diagnostic testing in mathematics outlines the main issues concerning the training of engineering students at ChNU. A rather small group of the involved engineering students can be related to the limiting factors at our university. At the same time, such project gives an opportunity to design and verify our diagnostic test as well as to find the main accompanying challenges.

Besides, we have identified some suggestions related to the procedure of developing diagnostic test in mathematics for engineering students, in general. Thus, the formation of a well-balanced bank of items which consists of both content-related questions in mathematics and behaviour-related tasks taking into account the needs of some engineering specialities is an immediate issue for the ChNU mathematics and engineering staff. Therefore, the results of the carried testing will be used to develop and approve the list of topics and to formulate the expected outcomes for test items by using the notions of mathematical competence.

It is necessary to support the information exchange between mathematics, physics, and engineering staff to reveal the common trends in mathematical education of future engineers. Moreover, the results of diagnostic testing in mathematics can be open for all the staff members. The implementation of diagnostic testing and the identification of so-called 'risk' students do not solve the problem of agreement of curriculum formal requirements and the actual level of students' preparedness concerning the low level of students' performance in some items. Thus, planning mathematics modules should include some hours intended to revise the topics learnt, therefore, the choice of the lesson content should be based on the diagnostic testing results.

CHAPTER SIX

CONTINUITY IN TEACHING MATHEMATICS

6.1. Continuity in Studying the Theory of Limits as a Pedagogical Problem

M. Bosovsky & A. Bozhko

Historical information about the formation of the theory of limits can be found in encyclopaedias[1] [2] (Bosovsky, 2005; Volkova, 2003), historiographical collections[3] [4] (Nikolsky, 1983; Fyhtenholts, 2001) and other referential and scientific literature. It should be noted that the authors of the XVII century had a very clear idea about the concept of the border consistency and convergence and considered it necessary to prove the convergence of the series.

The discovery of differential and integral calculus dates from the last third of the XVII century. Regarding publication, the priority of this discovery belongs to Leibniz[4] (1982), who gave a detailed summary of the main ideas of the new calculus in the papers published in 1686. As for the time of the actual receiving of the main results, there is every reason to believe that the priority belongs to Newton[5] (1989), who had come to the main ideas of differential and integral calculus in 1666. Widely known among the English mathematicians the "Analysis using equations with an infinite number of members" developed by Isaac Newton, who shared it with Barrow and Collins in 1669. Scientific approaches to the concepts of differential and integral calculus of Newton and Leibniz are different. Newton's basic concepts are the concepts of "fluents" (varying (flowing) quantity) and "fluxions "(instantaneous rate of change). He focused on the problem of fluxions relations and the relations between fluxions with specified fluents (compilation and differentiation of differential equations). Newton stated that the fundamental problems of the infinitesimal calculus were: (1) given a fluent (that would now be called a function), to find its fluxion (now called a derivative); and, (2) given a fluxion (a function), to find a corresponding fluent (an indefinite integral). However, neither fluent method and Newton's fluxions or differential calculus of Leibniz found the unanimous acceptance. That's why the mathematicians returned again to the study of fundamental concepts and principles of analysis.

The two theorems of limits by Jean d'Alembert[6] (1950) and S. Lyuilye[7] (1784) add a theorem border ratio of two variables and introduces the first border sign "lim"; the first is the derivative of a function in S. Lyuilye and it acts as a "differential ratio" (rapport differential) and is signed $\lim \frac{\Delta P}{\Delta x}$. The symbol $\frac{dP}{dx}$ is considered as a whole, and not a fraction. S. Lyuilye does not use the term "infinitesimal", keeping it relevant to refer to infinitesimal; and he had no concept of differential.

---

[1]Bosovsky, M.V. (2005). The theory of borders, for the implementation of continuity. Cherkasy University Journal, 74, 3-8. (in Ukr.).

[2]Volkova, N.P. (2003). Pedagogy: A Handbook for students. Higher teach. bookmark. Kiev: Publishing Center "Academy". (in Ukr.).

[3]Nikolsky, S.M. (1983). Course of mathematical analysis. Moscow: Nauka. (in Rus.).

[4]Leibniz, G.V. (1982). Works in four volumes. Moscow: Mysl. (in Rus.).

[5]Newton, I. (1989). The mathematical principles of natural philosophy. Moscow: Nauka. (in Rus.).

[6]Jean d'Alembert. (1950). *Dynamics*. Moscow-Leningrad: Gostekhizdat. (in Rus.).

[7]L'Huilier, S. (1784). Exposition élémentaire des calculs des principes supérieurs. Berlin: Berlin Académie des sciences.

The most essential point of boundary theories in the second half of the XVIII century was the refusal of the use of the infinitesimal algorithm of Leibniz[8] (1982). Long and nonlinear path of knowledge of the basic concepts and methods of the theory of boundary and its applications, and the results of a survey of pupils and students, give reason to believe that the acquisition of the modern theory by schoolchildren and students can be accompanied by considerable difficulties. Like it was for the mathematicians of the past, for today's young generation it is difficult to understand what the expression "sequence approximate values" means, as related to the operations on the approximate values of the same operations on their exact values, as in the same sequence numbers to determine whether it can be a sequence as an arbitrarily accurate approximation value of some magnitude more.

According to leading mathematicians[9] (Cherkasov, 1976), the formation theory of limits as a coherent mathematical theory today can be considered complete. The study of continuity in learning the theory of limits in secondary and higher education is a pedagogical problem.

Within the limits of the theory there are two relatively independent content blocks "Limit order" and "Limit function", and within them – a series of educational topics. We consider it necessary to enumerate them.

The content block "limit order": 1) the sequence and types; 2) border sequences; 3) the infinitely small and infinitely large sequence relationship between them; 4) properties border sequence; 5) monotonous sequence Weierstrass theorem and its applications; 6) euler's number, irrational number e; 7) limit points of the set and sequence; subsequence; 8) the upper and lower boundaries, Bolzano-Weierstrass theorem and its analogue; 9) convergence criterion of Cauchy sequences; 10) the principle of nested segments; another way to bring countless segment [0; 1].

Content block "limit function": 1) the definition of the border functions and their equivalence particular case of boundary features; 2) the uniqueness of the border; 3) Criterion existence of bilateral borders through unilateral; 4) the infinitely small and infinitely large functions. 5) transition in the border irregularities; 6) theorem intermediate function; 7) properties and applications of infinitesimal functions; 8) arithmetic operations on the borders; 9) the existence of borders monotonic function; 10) the criterion of Cauchy existence of borders for any function.

At present the theory of the boundary part of the course "Mathematical Analysis" and "Higher Mathematics" is taught to the students from different areas of training in classical, educational and other specialized (technical, technological, etc.) universities.

The studying of the theory of limits at the universities of Ukraine is regarded in varying degrees and with significant semantic differences. The deployment of the content theory of limits is exercised on four levels: purely deductive; partly deductive; based on deductive; intuitive and deductive.

Analysis of textbooks for the universities shows that learning the theory of limits can be divided into three relatively independent stages: propaedeutical, sampling, and direct.

The first phase relates to the formation of students' initial ideas about limiting transition. Introducing students to the idea of limiting transition takes place in the 6th form within the study of geometric concepts "circle", "square the circle". Here, on the visually-intuitive level the method of exhaustion, known to mathematicians since the 5 century, is used. Second, there follows the sampling phase of the study of the attribute elements of the theory of limits in the course of algebra and analysis of secondary

---

[8]Leibniz, G.V. (1982). Works in four volumes. Moscow: Mysl. (in Rus.).

[9]Cherkasov, R.S. (1976). About methodical preparation of math teacher in pedagogical institute. *Mathematics in school, 5,* 80-84. (in Rus.).

school. This measure of the content deployment of the element boundaries of the theory studied in non-core and core classes is very different. Direct learning stage of the theory of limits refers to the period of study at the university, where students learn mathematical analysis in the course of higher mathematics. But the concept phase of training theory of limits should be distinguished from the term "level study" of this theory. In terms of studying the theory of boundary we understand the degree of immersion in the theory of the student. The level of study, in our opinion, is

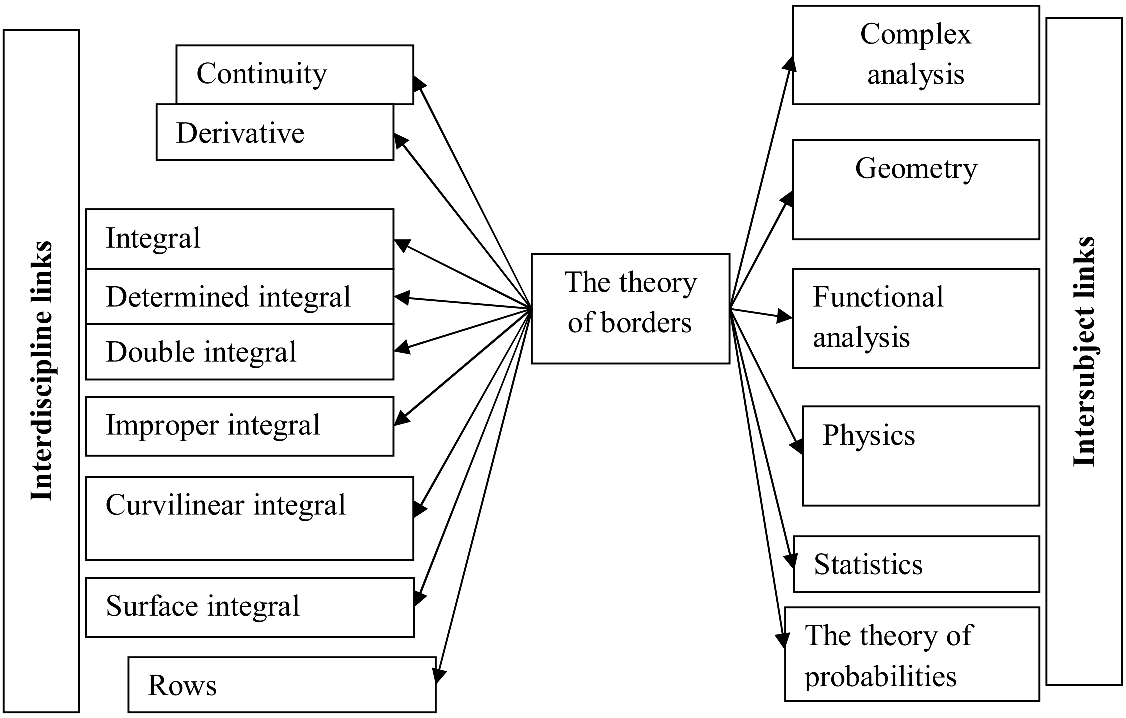


Figure 1. Application of the theory of borders

characterized by three components that should be considered as a unity. The first volume of the content being studied can be either bitmap (as in grade 6 (mathematics) or within the study of the geometry of basic school), fragmented (as in the course of algebra) or complete (as in the course of mathematical analysis in the university course) The second level of study describes the theory of limits through the number of members of the mathematical apparatus, which is used for the deployment of its content. It can be minimal, sufficient, or complete. The third characteristic of studying the theory of limits relates to the method of study (intuitive, user-deductive, or purely deductive), its basic concepts, and statements.

Theoretically, there exist 27 combinations of these characteristics and levels of study of the theory of limits. However, the practice of teaching shows that the real potential of these are only three: "intuitive", "infinitesimal", and "definitive."

The first of them is the intuitive level of the study of the theory of limits, it is characterized by: 1) "bitmap" amount of content; 2) minimum of mathematical tools; 3) no justification of facts; they replace the intuitive idea that does not contradict common sense. Thus, students unconsciously, instinctively are aware of the truth, make guesses based on the previous experience, knowledge, intuition, insight, guesses. This level is realized, as a rule, only at the propaedeutic phase of training, particularly in the 6th form.

Mathematical interpretation of the term "infinitesimal" is partly considered in the doctoral dissertation by V. Petrova[10] (1999). The idea of infinitesimal and of actual infinitesimal quantities goes back to the ancient era. Nowadays the infinitesimal concept is receiving an increasing attention within modern mathematics. Infinitely large and infinitely small numbers, mathematical atoms, "indivisible" monads are increasingly featured in various publications, are included in mathematical practice. The monads, according to the doctrine of Leibnitz, are understood as undestroyed, uncreative, immutable, but rather active structures that are part of the basic unit of all reality. With various organizational processes these basic units can be combined into complex units, each of which, due to its bound structure is Monad. In terms of the theory of informational and educational environment with the examples of infinitesimal methods necessary conditions for implementation of the learning process of transition from finite to infinite are described. Thus, the theory of the infinitesimal level boundary is studied if: 1) the extent of its content is fragmented; 2) using adequate mathematical apparatus; 3) in order to study the basic concepts and statements, intuition and common sense with the elements of deductive reasoning, are used.

This level of mastering the theory of limits can be implemented at all three stages of its study. At this level, the theory of limits is learned in the main school (Geometry, grade 9), in high school (Physics, grade 10; Mathematics, grades 10-11) and in the study of the higher mathematics course at university non-mathematical profile.

The third level study of the theory of limits – definitive- is characterized by: 1) it is the most complete; 2) comprehensive use of mathematical tools for deploying content; 3) purely deductive justification of basic concepts and statements.

The term "definitive" in this application according to its scientific understanding[11] (Raikov, 1982) is defined as clear, specific, clear and accurate. This level of the study of the theory of limits is implemented not only at the immediate stage of education (in universities, in natural mathematical and engineering areas), but also – in schools with the classes of the profound study of mathematics.

The relationship between the stages of study and levels of study of the theory of limits is shown in the table 1.

Table 1. Relationship between training stages and levels of study of the theory of limits

The levels of Study	**The levels of Study**
Propaedeutic	Intuitive
	Infinitesimal
Sampling	Infinitesimal
	Definitive
Direct	Infinitesimal
	Definitive

The term "level of study" of the theory of limits distinguishes the concept of the "level of assimilation" of this theory. In fact, the first concept relates to the objective characteristics of the learning process – educational content, and the second – to the

[10]Petrova, V.T. (1999). *Scientific and Methodological Fundamentals of learning mathematical disciplines in universities.* (Doctoral dissertation). Moscow: MHHHU. (in Rus.).

[11]Raikov, D.A. (1982). *One-dimensional mathematical analysis*. Moscow: Vysshaya Shkola. (in Rus.).

results of the subjective mastery of the content of those students. And pupils and students within the educational content can master it or use it at a reproductive level (simple reproduction of the information), or reconstructive variant level (with the ability to share the knowledge with the classmates), or creative levels when a new product is created subjectively in teaching and learning activities.

Observations show that on an intuitive level of the study of the theory of limits talking about the creative level of assimilation makes no sense. Similarly, the reconstructive variable level of assimilation is hardly possible. This explains the small amount of content, limited mathematical tools, and the character of the instruction used in training. In the study of the infinitesimal level pupils (students) are able to learn the theory of the limits (or elements) not only in the reproductive level, but also in the reconstructive variant. Each of the three levels of the assimilation of the limits can be achieved only at the definitive level of study.

In general, the concept of ensuring the continuity in studying of the theory of limits in universities should take into account the stages of study, level of study and levels of learning.

The relationship between levels of study and levels of the assimilation of the theory of limits is shown in the table 2.

Table 2. Levels of study and levels of assimilation theory of limits

Levels of study	Levels of assimilation
intuitive	reproductive
infinitesimal	reproductive
	reconstructive variable
definitive	reproductive
	reconstructive variable
	creative

The ensuring of the link between direct and selective stages plays a crucial role in learning, in the context of continuity, the theory of limits at the universities. At both stages the theory of limits can be studied at the infinitesimal (I) and the definitive (D) levels. This means that in the theory there are four basic circuit transitions from the study of the theory of limits at school to the study of the theory of limits in the university: 1) from the infinitesimal level in high school; 2) at the definitive level at the university; 3) from the infinitesimal level to the definitive level in universities; 4) from the definitive level to the infinitesimal level in universities. The first two schemes correspond to the evolutionary type of succession, the third – to the revolutionary one, and the fourth describes a setback in learning the theory of the limits. Students who studied mathematics in general become the university students of the non-mathematical profile. Those who studied in Physics and Mathematics classes are mostly students of pedagogical or technical universities. But not always. Sometimes the school leavers with the school "average" grade in Mathematics and Physics become the students of Mathematics and Physics. Thus, the fact that they are lacking in the mathematical development, is not always taken into account. There is a gap in the study of the theory of limits.

The biggest gap in studying the theory of limits is observed in the transition I – D. This gap is explained by social factors. Students with low grades in exact sciences enter the Institutes and Faculties where the exact sciences are a major. Therefore, the main focus lies on ensuring the transition and the continuity of learning the theory of the limits.

The analysis of textbooks[12] [13] [14] (Shkil, Slyepkan & Dubynchuk, 2006; Kolmogorov, Abramov, Dudnitsyn, Ivlev & Shvartsburd 1995; Bevz, 1995) and tutorials[15] [16] (Akulenko, Klyatska & Ous, 2007; Verbytsky, 1991) for high schools showed that the theory of limits can be deployed in the content of education at the following levels: purely deductive, part-deductive, based on deductive, and intuitive and deductive.

The first two levels of the deployment of the content provide a definitive theory of limits, and the rest – the infinitesimal one. So, there are two kinds of transition I – D. The first of them (D) is associated with a situation where a non-prepared school leaver entered the Physics and Mathematics classical or pedagogical university and will study the theory of limits in the largest amount and the highest level of rigor – in the course of mathematical analysis. The second type of transition (I) is typical for the situations where the unprepared school leavers taking a University specialty of engineering and study the course of "Higher Mathematics" where the theory of the limits is taught on a partly-deductive level. Both transitions require detailed attention in the context of the continuity of learning. The implementation of the continuity between school and university courses in mathematics can be realized in two ways. The first one is based on a new meaning of the content mastered at the previous stage of training, and then there is a link of what is studied at the university with the material that was studied at school. Therefore, the first direction is characterized by combining repetition and continuity. The second occurs when the content at this stage of training prepares students for successful mastery of the material at a later stage, that there is a connection of the material that was taught at schools, with the material that needs to be reviewed at the university. Then comes the continuity of the functions prospects.

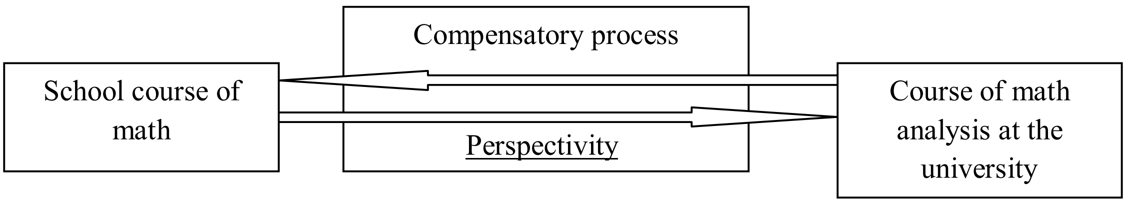


Figure 2. Areas to ensure continuity

The choice depends on what is defined as an object of study: the process of learning mathematical analysis course at the university or the mathematics learning process at school. In the first case the ensuring of the continuity requires repetition. During the study course, different parts are connected with the school math courses, a periodic reviewing of the previously learned information is important for establishing the lasting memory and associative relationships between the concepts studied and those which have been previously fixed in memory. Regular checking of the information stored in a long-term memory of the student, allowing the time to identify and eliminate distortions fixed in memory is of importance. The distortion is usually expressed in the misunderstanding of the concepts and their relationships, it happens because of various reasons. The first cause may be in the student as, despite the impeccable presentation,

[12]Shkil, M.I., Slyepkan, S.I., & Dubynchuk, O.S. (2006). *Algebra and Introduction into Analysis. Grade 11: Tutorial*. Kiev: Zodiac – ECO. (in Ukr.).

[13]Kolmogorov, A.M., Abramov, O.M., Dudnitsyn, Y.P., Ivlev, B. M. & Shvartsburd, S. I. (1992). *Algebra and Introduction into Analysis. 10-11 grades*. Kiev: Osvita. (in Ukr.).

[14]Bevz, G.P. (1995). *Mathematics. Grade 11*. Kiev: Osvita. (in Ukr.).

[15]Akulenko, I.A., Klyatska, L.M. & Ous, I.V. (2007). Algebra and Number Theory: Teaching methodical guide for organizing classroom and independent work. Part I. The theory of divisibility in the ring of integers. Cherkasy: Ed. from. Cherkasy National University named after Bohdan Khmelnytsky. (in Ukr.).

[16]Verbytsky, A.A. (1991). Active learning in High School: context approach: educational tutorial. Moscow: Vysshaya Shkola. (in Rus.).

they might form distorted concepts studying the new material or because of the absence of the general scientific (philosophical) understanding of the concepts of life that are not always consistent. Secondly, the causes might not depend on the student personally – when the presentation of the material was inaccurate, for example, if not all mathematical concepts were formulated on the proper scientific level.

According to N. A. Tarasenkova[17] (2002), a personal training aspect in the context of semiotic approach to mathematics education, the implementation of semiotic approach to mathematics education is associated with the problems of methods of teaching, where the main emphasis is put on the communication objectives, content, methods, tools, and organization of learning the structure and functioning of sign systems in learners and which relates the semiosis with the educational process. From the standpoint of this approach, the students, while studying mathematics, have to arrange a purposeful process of forming the functional semiotic systems. During training, the selection and use of semantic-symbolic membranes should be based on the analysis of the conflict between the logical and the visual, which can be not only of objective, historically conditioned character, but may originate in subjective reasons – lack of understanding of the origin and content of the training material, negative attitudes to the ability to understand the content of this; inability to wrap the content in different semantic-symbolic shells; the presence of adhesions (absence of dialectical connections) of content and form formed in the experience of students during the previous studies, and more. The application of the uniformity in teaching mathematics leads to the inhibition of the process of perception and processing of the data by the students and prevents the deployment of the full content of the educational material and ultimately negatively affects learning goals.

Overall, the system that will ensure the continuity of the learning of the theory of limits should include the following components: personal, content, semiotic, and organizational (methods, forms, and means of education). The first component is the student as the central figure in the educational process. However, the personal component not only affects the two other components of the system, but also depends on them. It is therefore important to know the features of the selection of the content, methods, forms, and means of teaching the theory of limits in the context of continuity.

In teaching the theory of limits on any of its stages (propaedeutic, sampling, and direct) and any level of study (intuitive, infinitesimal, and definitive) the students at first are to perceive, comprehend, and memorize the new learning material (first level of learning).

The specifics of teaching mathematics, in particular – the theory of limits, need special testing of knowledge and skills of the students (pupils) and forming the respective competences. Only by solving problems and doing training exercises the students can move on to the second-level learning, that is – apply knowledge in practice according to the patterns shown by the teacher. Consequently, the use of reproductive techniques is objectively necessary in studying the theory of limits, regardless of the stage of training and study.

The third level of learning methods is associated with developing the creative activity of the students. For this purpose, the generally applicable methods are: of problem exposition, part-searching, and research methods. And if in teaching the theory of the limits the method of presentation of the problem is a must, the choice of two other methods should correlate with the amount of the training content and its complexity, peculiarities of the learning environment, and the time factor.

So, we have experimentally set the provisions in the selection of teaching methods aimed at ensuring the continuity in learning the theory of limits:

---

[17]Tarasenkova, N. A. (2002). *Using sign and symbolic means in teaching mathematics. Monograph*. Cherkasy: "Vidlunnya-Plyus". (in Ukr.).

1) contents of the theory of limits are considerably complicated and should be studied at three levels: intuitive, infinitesimal, and definitive;

2) some specifications of the teaching of the theory of limits and the levels of its assimilation determine the specific ways of learning and their variability;

3) any method of learning may be used at each level of study of the theory of limits;

4) the more detailed character the instructional presentation of the content of the theory of limits has, the more time and effort is needed to organize its assimilation by the students (pupils) as each level of complexity piles up the learning material (an objective factor) and the difficulty in mastering his (a subjective factor);

5) increase of the content of the theory of limits (with the accompanying objective and subjective factors), as well as the time factor lead to fewer variations in the choice of the methods of learning, which can be applied in the actual educational process.

CHAPTER SEVEN

COMPARATIVE STUDIES

7.1. Retrospective of the Development of School Mathematics Education in the Second Half of the XXth Century and Until Now

N. Tarasenkova & Z. Serdiuk

The development of a society is priori impossible without certain changes in its life spheres including education. The reform of school mathematics education has a special role in this context.

The reform of school education, including mathematical, and integration into the world education community require from the modern school to solve the urgent tasks of improving the quality of school education. To implement the tasks set in the State Target Program of Improving the Quality of School Natural Sciences and Mathematics Education until 2015, the State Standard of Secondary Education (second generation) and other public documents, it is important to study the experience of the world's best education systems, including the state, trends and patterns of education development in different countries, geopolitical regions and in the world as a whole. It is this aspect that is the study subject of comparative pedagogy as a branch of pedagogical science. The use of basic conceptual principles of comparative pedagogy for the research of trends and the development of national and foreign school mathematics education will enable to expand the boundaries of cross-sector analysis and synthesis and to get new data for improving the system of mathematics education in Ukraine.

We focus our attention on three periods of reforming the content of school mathematics education in the second half of the XX century and the beginning of the XXI century that are associated with the direct participation of the scientists of the Bohdan Khmelnytsky National University at Cherkasy in creating school programs and textbooks for secondary schools.

The period of Kolmogorov's reform (60-s – 70-s of the XX century). The reform of school mathematics education in the 60-70-s has a personalized name of Academician A. Kolmogorov, a leading mathematician of the world at that time, a pioneer, chief inspirer and direct executor of theoretical and practical reform measures. The main objective of Kolmogorov's reform was to modernize the content of teaching Mathematics, to bring it closer to the problems of modern science and, to some extent, to move further away from the classic issues having been considered since ancient times. In particular, the following measures were planned: algebrazation of Mathematics course in junior school, introduction of differential and integral calculus and probability theory in senior school, the complete reconstruction of Geometry course based on geo-metric transformations and vectors. Therefore, the ways to implement these ideas in contemporary programs and textbooks, especially in Geometry, deserve special attention.

The analysis of the recent research and publications shows that the interest to the phenomenon of Kolmogorov's reform is not reduced. Currently, this period of reforms is a subject of both historical-pedagogical research and the study in the field of didactics and methodology of teaching Mathematics: A. Abramov[1] (2003), D. Dobrov[2] (2012),

[1]Abramov, A. M. (2003). *On the status of mathematics education in high school (1978-2003).* – Moscow: Fazis (in Rus.).

[2]Dobrov, D. (2012). Teaching of Mathematics in school and Kolmogorov's reform: [Online]. Available: http://www.dm-dobrov.ru/history/mathematics.html (in Rus.).

T. Kiseleva[3] (2006), S. Kogalovskii[4] (2006), Y. Kolyagin[5] (2001), G. Kondratieva[6] (2009), I. Kostenko[7 8] (2012 & 2014), A. Stolyar[9] (1986), V. Testov[10 11] (2011 & 2015), B. Furtak & D. Zhyvko[12] (2000), V. Zuckerman[13] (2012), R. Cherkasov[14] (1997), etc. However, the scientific exploration of predecessors is not exhaustive in this direction. They can and should be continued with the use of interfield analysis and synthesis, in particular, on the basis of a new direction of theoretical and methodological research in the subject area of "mathematics" – comparative didactics of mathematics[15] (Tarasenkova & Serdyuk, 2013).

According to the periodization of G. Kondratieva[16] (2012), Kolmogorov's reform in the time of dimension covers two phases (reformist and experimental-eclectic) of new cycle of the development of school mathematics education in the former state, of the second half of the twentieth century.

The beginning of the reform phase (1965) conformed with a significant event of that time – founding the Central Commission of Science Academy and Pedagogical Scien-ce Academy of the USSR and determining the content of secondary mathematical education (Chairman – Academician A.I. Markushevych, supervisor and simultaneously the Chairman of the Mathematics Scientific Methodological Council of the USSR Education Ministry – Academician A.M. Kolmogorov).

In 1968, the new program in Mathematics for secondary school was approved and the work on the creation of new textbooks began. In 1970, "The Regulations about Secondary School" was adopted to mark the beginning of the next experimental-eclectic phase of the reform, in the opinion of G.V. Kondratieva[17] (2009).

According to the new curriculums and programs:

- the primary school became a three-year course; "Mathematics" course was

[3]Kiseleva T. (2006). The problem of periodization in the studies on the history of mathematical education. *Bulletin of Eletski State University named after I.A. Bunin. A series of "History and theory of mathematics education", 11,* 132-140 (in Rus.).

[4]Kogalovskii, S.R. (2006). On the leading plans of teaching mathematics. *Pedagogika (Pedagogy), 1,* 39-48 (in Rus.).

[5]Kolyagin, Y.M. (2001). *Russian school and mathematics education: Our pride is our pain*. Moscow: Prosveshcheniye (in Rus.).

[6]Kondratieva, G.V. (2009). On the problem of the periodization of the development of school mathematics education in Russia. *Bulletin of Moscow State Regional University. "Pedagogy" series, 3,* 124-131 (in Rus.).

[7]Kostenko, I.P. (2012). Reform of school mathematics in 1970-1978 years. On the 40th anniversary of the «Kolmogorov reform». *«Alma mater» (Bulletin of the Higher School)*: [Online]. Available: https://almavest.ru/ru/node/1256 (in Rus.).

[8]Kostenko, I.P. (2014). 1965 – 1970 years. Organizational preparation of the reform-70 ME, APS, personnel, programs, textbooks (Article Five). *Matematicheskoye obrazovaniye (Math education), 3(71),* 2–18 (in Rus.).

[9]Stolyar, A. (1986). *Psychology of Mathematics: [lectures]*. Minsk: Vysheyshaya School (in Rus.).

[10]Testov, V.A. (2011). Selection of content of teaching mathematics. *Mathematical methods and models: theory, applications and role in education: collection of scientific works*, 265-273 (in Rus.).

[11]Testov, V.A. (2015). The main aspects of the implementation of the concept of development of mathematics education in the Russian Federation. *Innovative projects and programs in education, 1*: [Online]. Available: Http://cyberleninka.ru/article/n/osnovnye-aspekty-realizatsii-kontseptsii-razvitiya-matematicheskogo-obrazovaniya-v-rossiyskoy-federatsii (in Rus.).

[12]Furtak, B. & Zhyvko D. (2000). New approaches to the content of mathematics education in Ukraine. *Matematyka v shkoli (Mathematics in school), 5,* 24-30 (in Ukr.).

[13]Zuckerman, V. (2012). Kolmogorov's Reforma and school mathematics education today. *Library "Moscow State University for School"* : [Online]. Available: http://lib.teacher.msu.ru/pub/2330 (in Rus.).

[14]Cherkasov, R.S. (1997). The history of national school math education. *Matematika v shkole (Mathematics at school), 4, 5, 6* (in Rus.).

[15]Tarasenkova, N.A., Serdyuk, Z.O. (2013). The basis of comparative pedagogy in mathematical education of different countries investigation. *Didactics of mathematics: problems and research: international collection of scientific works*, 40, 55–59 (in Ukr.).

[16]Kostenko, I.P. (2012). Reform of school mathematics in 1970-1978 years. On the 40th anniversary of the «Kolmogorov reform». *«Alma mater» (Bulletin of the Higher School)*: [Online]. Available: https://almavest.ru/ru/node/1256

[17]Kondratieva, G.V. (2009). On the problem of the periodization of the development of school mathematics education in Russia. *Bulletin of Moscow State Regional University. "Pedagogy" series, 3,* 124-131 (in Rus.).

introduced replacing Arithmetic (*now, primary school is a four-year course);*

- the structure and the names of the systematic course components in Mathematics: grades 4-5 – propaedeutic course of "Mathematics" with elements of Algebra and Geometry; grades 6-8 – systematic courses of "Algebra" and "Geometry" (Planimetry), grades 9-10 – systematic courses of "Algebra and Analysis Introduction" and "Geometry" (Stereometry) (*the changes have been preserved)*;

- the construction of the course became linear-concentric (*the changes have been preserved)*;

- set Theory and Mathematical Logic acquired the status of fundamental principles and were used as the primary language of the content (*the changes have not been preserved);*

- archaic issues were removed from the courses (for example, the algorithm of extracting the square root of the number) (*the changes have been preserved*);

- the program included some questions about computers and programming *(the changes have not been preserved, but in the modern curriculum of secondary education, there is a discipline "Informatics");*

- a new form of education, optional courses, were offered; they were "Related Sections and Questions of Mathematics" (the expansion of some program themes) and "Selected Issues of Mathematics" (programming, computational mathematics, vector algebra, linear programming problems) (*the changes have been preserved and developed*);

- the mark for mastering optional courses gained official status and was recorded in the secondary education certificate (*the changes have not been preserved*);

- a system of schools and classes with advanced theoretical and practical study of certain subjects, including Mathematics, which was launched in 1959, in 1966 was completed with Physics-Mathematics boarding schools that were created at leading universities (*the changes have been preserved*).

Trial textbooks in Geometry for grades 6, 7 and 8 by O.F. Semenovych[18] [19] [20] (1961; 1962; 1963) were published in 1961-1963, three years before approving the programs. Those textbooks were created based on the current programs (not Kolmogorov's one); however, according to the Academicians V.M. Glushkov and S.M. Chernikov[21] (2010, by Preprint, 1970), those textbooks reflected the new tendencies that later formed the basis for new programs in the secondary school Geometry. On the basis of these books, the author team consisting of O.F. Semenovych, F.F. Nagibin and R.S. Cherkasov developed and submitted to the contest the textbook of Geometry for 8-year school, where it received the second prize of the Ministry of Education of the RSFSR (the first prize was not awarded to any textbook). A.M. Kolmogorov highly appreciated the authors' attempt to form the school course in Geometry on theory-set basis with wide involvement of geometric transformations and vectors and invited those authors to cooperate. The author team headed by A.M. Kolmogorov was created and worked to form and improve the textbook in Geometry for grades 6, 7 and 8 in the middle of 60-s – early 80-s. The textbooks were published in separate books and were tested in experimental study in the respective forms of the national secondary schools. Experimental data were collected every year, on the basis of which the author team made adjustments in the textbooks. About 10 samples of the trial textbooks were

[18]Semenovich, A.F. (1961). *Geometry. Trial textbook for the sixth grade*. Ulyanovsk, 164 p. (in Rus.).

[19]Semenovich, A.F. (1962). *Geometry. Trial textbook for the seventh grade*. Ulyanovsk, 94 p. (in Rus.).

[20]Semenovich, A.F. (1963). *Geometry. Trial textbook for the eighth grade*. Ulyanovsk, 90 p. (in Rus.).

[21]Glushkov, V.M., Chernikov, S.N. (2010). Review about scientific and pedagogical activity of head of the department of geometry and mathematics teaching methods Cherkasy Pedagogical Institute, candidate of physical and mathematical sciences, associate professor Semenovich A.F.: [Preprint, 1970]. *International scientific and methodical conference "Problems of mathematical education" (PME – 2010), Cherkasy, 24-26 November 2010* (in Rus.).

published for each grade. The top of this process was the textbook "Geometry 6-8"[22] (1979), which was published in almost 20 languages.

We point out that the creative contribution of O.F. Semenovych is 150 published works, 81 of which are separate editions. These works focus on the problems: 1) fundamentals of Geometry and geometric constructions in hyperbolic plane; 2) teaching Geometry in the pedagogical institutes, particularly, 12 books in Projective Geometry and the Fundamentals of Geometry; 3) teaching Mathematics in secondary school: the textbooks in Geometry, teachers' books, books and articles for schoolchildren, students of pedagogical institutes, teachers. Most works by O. Fedorovych have not lost their value now.

We characterize the structure-content features of plane geometry course implemented in this textbook.

Learning material for each grade is presented in some program themes (they are presented as chapters in the textbook)[7] including:

- in the 6th grade, the themes are "Elementary Notions of Geometry", Congruence of Figures and Motion";
- in the 7th grade, the themes are "Parallelism and Parallel Transfer", "Polygon", "Vector", Similarity";
- in the 8th grade, the themes are "Turns and Trigonometric Functions", "Metric Relations in a Triangle", "Incircle and Excircle Polygons", "Initial Information in Stereometry".

Each program theme is divided into paragraphs, which are divided into items. The exception is the first and the fifth themes, in which the division of the learning material into paragraphs is not provided; however, there are from 7 (theme "Vectors") till 14 (theme "Elementary Nprogram theme otions of Geometry") items here. In general, there are 100 items in the textbook. Every program theme involves an item (not always the last one) with additional (extracurricular) material. At the end of each paragraph, there is an item (without a number) "Additional Tasks to the Paragraph". At the end of the textbook before appendices, there are "Tasks to test course knowledge of grades 6-8" and answers. Some additional information and reference materials, e.g. "About Logical Construction of Geometry", "Language of Set Theory in Geometry", are placed at ten pages of "Appendices".

There are theoretical materials, questions and tasks to develop students' knowledge and skills in the structure of each item. The amount of learning content is small and designed mostly for one lesson. The questions on theoretical material and tasks are not separated but intertwined forming one unit. Clear and hidden differentiation of tasks according to the degree of their difficulty is used in the sets of tasks. The tasks for oral solving are marked with a "nought" at the number of a task; the tasks of high complexity are marked with a "star". Internal differentiation is realized through increasing the complexity of the tasks hidden for students – each next task is more complex than the previous one according to the content and operationally.

We should point out that in the modern Ukrainian textbooks in Geometry, more detailed sections are applied and other sections are offered[23] (Kolmogorov, Semenovich, & Cherkasov, 1979). For example, in the textbook of Geometry for grade 7[24] (Burda & Tarasenkova, 2007) (corresponds to grade 6 in 60-s – 70-s of the XX century), there are four chapters with learning material and the following separate sections of the book: opening remarks to the students, summary tables and tasks to repeat the studied material

---

[22]Kolmogorov, A.N., Semenovich, A.F., Cherkasov, R.S. (1979). *Geometry: Textbook for 6-8 grades of the secondary school*. Moscow: Prosveshcheniye, 382 p. (in Rus.).

[23]Kolmogorov, A.N., Semenovich, A.F., Cherkasov, R.S. (1979). *Geometry: Textbook for 6-8 grades of the secondary school*. Moscow: Prosveshcheniye, 382 p. (in Rus.).

[24]Burda, M.I., Tarasenkova, N.A. (2007). *Geometry. Textbook for the 7th grade of the secondary school.* Kyiv: Publishing House "Osvita", 208 p. (in Ukr.).

at the end of the school year, answers and subject index. Each chapter starts with a heading “In the Chapter you will learn“, the instructional material is placed in some paragraphs, and a chapter is completed with test questions and test tasks. Each paragraph contains learning material that students have to learn, additional information (section “Learn More”), test questions (section “General Recall”), and the tasks for practicing skills (section “Solve the problem”).

The section “Learn more” includes: interesting material of the theme to be learnt, the related material of extracurricular character; the data of names and symbols origin; historical information; biographies of outstanding national and foreign mathematicians. The main unit of the tasks to a paragraph contains the tasks of four complexity levels. The first level is indicated by a stroke; they are largely oral tasks. The second level is marked with a nought; it is a compulsory task for training basic skills. The third level has no signs. These tasks correspond to sufficient level of students’ academic achievements. The fourth level is marked with a star; they are the tasks of high complexity level that allow students to demonstrate their mathematical abilities. In the textbook, the tasks are given in a traditional text form and in the forms of tables and are based on drawings. Some problems of a paragraph are in bold type. They are basic tasks (task-facts). Students should remember their formulation. These geometric statements can be applied to solving other problems. The methods of solving problems are given in not only the text of the paragraph, but in the unit of tasks for training skills. In addition to the main unit, the tasks of practical character are offered to each paragraph (section “Apply in practice”).

Thus, the apparatus for the organization of mastering and the apparatus of orientation in modern textbooks is more descriptive and detailed compared to the textbooks of the time of Kolmogorov’s reform. Let us consider the peculiarities of the contents disclosure of several initial items of the first program themes “Elementary Notions of Geometry” in the textbook[25] (Kolmogorov, Semenovich, & Cherkasov, 1979). As we have noted, the authors identified 14 items in this theme, including:

1) what is the geometric figure;
2) the basic concepts that are accepted without definitions;
3) magnitudes and numbers;
4) the basic properties of distances;
5) the relative position of three points on the line, triangle inequality;
6) a line segment and a ray;
7) coordinates on the line;
8) polygonal chain;
9) plain, planimetrics;
10) range;
11) polygon;
12) half-plane angle;
13) relative position of two circles;
14) from the history of geometry.

The main initial ideas of the author's conception of a content construction and its deployment in the book become apparent when considering the first three points of the first section.

The period of reforms in 80-s – 90-s of the XX century. In this period, there was a tendency to unload the content of mathematical education and to increase its practical focus that was reflected in new programs in Mathematics for secondary schools of 1980 and 1985. In these programs, the changes concerned not only the content of education but the structure of the program (including the new chapters “The organization of

[25]Kolmogorov, A.N., Semenovich, A.F., Cherkasov, R.S. (1979). *Geometry: Textbook for 6-8 grades of the secondary school*. Moscow: Prosveshcheniye, 382 p. (in Rus.).

educational process", "Guidelines for knowledge assessment", "Intersubject links", etc.). During the reforms of that period, 1989 became a turning point to some extent due to the adoption of a new Concept of school mathematics education. The ideas of humanization and humanitarization of education including mathematical education acquired the status of central ones. According to the idea of humanization, mathematics had to become rather a way of the existence and general culture of every citizen of a society than the goal of study; and its study had to provide an equal opportunity for every student to develop their personalities. According to another conceptual idea, differentiated approach was the basis of reforming educational branch "Mathematics" in secondary education; attention to the organization of multilevel study of the course was intensified within both various profiles of training the students of senior school and within one profile. Invariant component (in the classroom – within the curriculum) and variant (in extracurricular time – through optional classes, mathematical circles, extramural mathematical schools, individual and group work with gifted students, etc.) component of education were distinguished. Great attention was paid to school mathematical education of "high level". The advanced study of Mathematics became possible since grade 8.

In this period, there was a peak of creative activity of Kovalenko Volodymyr Gavrilovic (1932-1994), a Candidate of Pedagogical Sciences, Assistant Professor, the Honored Teacher of the USSR, the Head of Geometry and Mathematics Teaching Methods the Department (Mathematics Department since 1987) of the Cherkasy National University, in scientific research. His teacher guides "Mathematical Symbols" (co-author I.F. Sledzynsky, ed. I.F. Teslenko. – K.: Radyans'ka Shkola, 1981), "Problem Approach to Teaching Mathematics" (co-author I.F. Teslenko. – K.: Radyans'ka Shkola, 1985), "Didactic Games at Mathematics Lesson" (M.: Prosveshchenie, 1990); student's and teacher's book "Geometric Transformation of a Plane" (co-author O.F. Semenovych. – K.: Vyshcha Shkola, 1993. – A series of "Library of Physical and Mathematical School. Mathematics") were published. V.G. Kovalenko described his ideas and the obtained results in a number of articles, materials and abstracts.

However, the main achievement of V.G. Kovalenko was and has been the first (in Ukraine) experimental textbooks for the advanced learning of Algebra in grade 8[26] (Kovalenko, Krivosheev & Lemberskyy, 1991) and grade 9[27] (Kovalenko, Krivosheev & Lemberskyy, 1992). These textbooks were created together with well-known, in Cherkasy Region, experienced Mathematics teachers V.Y. Kryvosheiev and L.Y. Lembersky, and with an expert in IT-sphere O.V. Staroseltseva. Nowadays, many teachers use these books as a source of additional information and successful set of exercises and tasks.

The modern stage of reforming a school mathematical education is characterized by numerous changes in both the main conceptual principles of the reforms and the ways of their implementation in practice that requires special discussion and is beyond the scope of this article. However, the participation of scientists of the Cherkasy National University in the development of the content of school mathematical education cannot be overlooked. We can affirm that a group of associates, who dedicated themselves to the development of school mathematical education in Ukraine, was formed at the Physics and Mathematics Department of the Cherkasy National University (ChNU). The textbooks of this writing team became the winners of the all-Ukrainian competition of school textbooks including

[26]Kovalenko, V.G., Krivosheev, V.Y., Lemberskyy, L.Y. (1991). *Algebra. 8th grade: experimental textbook for schools with advanced study of mathematics and special schools of Physics and Mathematics profile.* Kyiv, education «Osvita», 302 p. (in Ukr.).

[27]Kovalenko, V.G., Krivosheev, V.Y., Staroseltseva, A.V. (1992). *Algebra. 8th grade: experimental textbook for schools with advanced study of mathematics and special schools of Physics and Mathematics profile.* Kyiv, education «Osvita», 271 p. (in Ukr.).

the textbook in Mathematics for grade 5[28] (Tarasenkova, Bochko, Bogatyreva, Kolomiets & Serdiuk, 2013) and grade 6 [29] (Tarasenkova, Bogatyreva, Kolomiets & Serdiuk, 2014), the textbook in Algebra for grade 7[30] (Tarasenkova, Bogatyreva, Kolomiets & Serdiuk, 2015) and grade 8[31] (Tarasenkova, Bogatyreva, Kolomiets & Serdiuk, 2015), the integrated textbook in Geometry for grade 11 (academic and profile levels)[32] (Burda, Tarasenkova, Bogatyreva, Kolomiets & Serdiuk, 2013). Didactic support in the form of a series of teaching and learning aids for teachers and students of secondary schools was developed for each textbook. Totally, over 100 textbooks were published including 70 ones recommended by the Ministry of Education and Science of Ukraine.

In 2015, the writing team including N.A. Tarasenkova (ChNU), M.I. Burda (the Institute of Pedagogics of Ukraine National Academy of Pedagogical Sciences), O.I. Hlobin (the Institute of Pedagogics of the National Academy of Pedagogical Sciences of Ukraine), I.M. Bohatyreva (ChNU), O.M. Kolomiets (ChNU), Z.O. Serdiuk (ChNU) developed a concept and innovative set of textbooks of a series “Test of Subject Competences” in Mathematics for grade 5[33] (Tarasenkova, Bogatyreva, Kolomiets & Serdiuk, 2015) and grade 6[34] (Tarasenkova, Bogatyreva, Kolomiets & Serdiuk, 2015), in Algebra for grade 7 [35] (Tarasenkova, Hlobin, Bogatyreva, Kolomiets & Serdiuk, 2015) and Geometry for grade 7[36] (Tarasenkova, Burda, Bogatyreva, Kolomiets & Serdiuk, 2015). It may be considered to be the beginning of practical stage of reforming school mathematical education in Ukraine on the basis of competence approach and gradual approximation of Ukrainian system of mathematical education to European and world standards.

It should be noted that in Ukrainian textbooks, the learning content associated with values and their properties is given exclusively inductively on examples using description, demonstration and characteristics. Other assumptions of the fundamentals of school Geometry course is also revealed inductively. In general, modern textbooks provide available learning content more fully; the ways of its presentation implement their correspondence to the age peculiarities of students more integral and may-sided.

Conclusions. The application of methods of comparative didactics of mathematics including those associated with its time (retrospective) vector allows to make a conclusion. The reform of school mathematical education of 60-s – 70-s of the XX centuries made fundamental changes to curricula, programs and textbooks in Mathematics of secondary school. Most changes have been preserved till now; some of them have been further developed. The content innovations of school course in Geometry of Kolmogorov’s reform time did not stand the test of time and are not appropriate nowadays.

---

[28]Tarasenkova, N.A., Bochko, O.P., Bogatyreva, I.M, Kolomiets, O.M., Serdiuk, Z.O. (2013). *Mathematics. Textbook for the 5th grade of the secondary school.* Kyiv: Publishing House "Osvita", 352 p. (in Ukr.).

[29]Tarasenkova, N.A., Bogatyreva, I.M, Kolomiets, O.M., Serdiuk, Z.O. (2014). *Mathematics. Textbook for the 6th grade of the secondary school.* Kyiv: Publishing House "Osvita", 304 p. (in Ukr.).

[30]Tarasenkova, N.A., Bogatyreva, I.M, Kolomiets, O.M., Serdiuk, Z.O. (2015). *Algebra. Textbook for the 7th grade of the secondary school.* Kyiv: Publishing House "Osvita", 304 p. (in Ukr.).

[31]Tarasenkova, N.A., Bogatyreva, I.M, Kolomiets, O.M., Serdiuk, Z.O. (2016). *Algebra. Textbook for the 8th grade of the secondary school.* Kyiv: UOVTS "Oríon", 336 p. (in Ukr.).

[32]Burda, M.I., Tarasenkova, N.A., Bogatyreva, I.M, Kolomiets, O.M., Serdiuk, Z.O. (2013). *Geometry. Textbook for the 11th grade of the secondary school.* Kyiv: Publishing House "Osvita", 304 p. (in Ukr.).

[33]Tarasenkova, N.A., Bogatyreva, I.M, Kolomiets, O.M., Serdiuk, Z.O. (2015). *Test subject competences. Mathematics, 5th grade of the secondary school.* Collection of tasks for assessment of pupil achievements [Teach. method. guidances.]. Kyiv : «Orion», 48 p. (in Ukr.).

[34]Tarasenkova, N.A., Bogatyreva, I.M, Kolomiets, O.M., Serdiuk, Z.O. (2015). *Test subject competences. Mathematics, 6th grade of the secondary school.* Collection of tasks for assessment of pupil achievements [Teach. method. guidances.]. Kyiv : «Orion», 40 p. (in Ukr.).

[35]Tarasenkova, N.A., Bogatyreva, I.M, Globin, O.I., Kolomiets, O.M., Serdiuk, Z.O. (2015). *Test subject competences. Algebra, 7th grade of the secondary school.* Collection of tasks for assessment of pupil achievements [Teach. method. guidances.]. Kyiv : «Orion», 32 p. (in Ukr.).

[36]Tarasenkova, N.A., Burda, M.I., Bogatyreva, I.M, Kolomiets, O.M., Serdiuk, Z.O. (2015). *Test subject competences. Geometry, 7th grade of the secondary school.* Collection of tasks for assessment of pupil achievements [Teach. method. guidances.]. Kyiv : «Orion», 24 p. (in Ukr.).

7.2. Implementing the Integrative Relations of Mathematics in Extra-Curricular Activities: Foreign Experience

M. Bosovskyi & M. Donets

The Law of Ukraine «On General Secondary Education»[1] (1999), Regulations about Comprehensive Educational Institutions[2] (2010) focus on the fact that learning, education, and development of pupils in comprehensive educational institutions is carried out at the usual hours, outside regular hours and during extra-curriculum activities. The efficiently organized class work is equally important and is aimed at building a fully developed, harmonious personality of a pupil. In terms of modernization of the system of general secondary education it is necessary to search for and implement new approaches to the activities performed outside regular hours. Implementing an integrative approach in the educational process of a comprehensive school provides for the establishment of meaningful interdisciplinary relations, interpenetration of knowledge and techniques from different disciplines; ensures diversification of the methods and organizational forms of learning; helps enhance pupils' thinking and the formation of their scientific outlook. Organization of extra-class activities in mathematics based on an integrative approach enables increasing pupils' interest to studying mathematics; awareness of its role in the cognition of reality, the need for assimilating mathematical knowledge as an integral part of human culture; creation of a coherent world view of pupils.

Integrating modern education system in the European educational space necessitates studying foreign experience to develop and implement mechanisms for realization of integrative relations of mathematics in the process of extra-class activities.

The analysis of research papers of foreign researchers[3] [4] [5] (Boaler, 2000; Pruski & Fridman, 2014; Ward-Penny, 2011; etc.) shows that the realization of the integrated approach in the educational process of a comprehensive school, particularly in learning mathematics, is carried out at the interdiscipline (focus on acquiring knowledge learned on other subjects) and interdisciplinary (integration of knowledge on various subjects for the solution of a problem) levels. The idea of H. Dodd[6] (1991) is valuable for our research, he notes that the integrated approach provides for not just combining or supplementing elements of learning (topics, methods), but also solutions to the problems that cannot be solved through a single subject. The team of authors led by J. Johnston[7] (2014) believes that the introduction of integrative approach ensures obtaining systematic knowledge by pupils and development of skills to transfer knowledge obtained to the situations in everyday life.

It is worth noting that foreign researchers consider the realization of integrative relations of mathematics with related subjects, such as those of natural or technological

---

[1]Verkhovna Rada of Ukraine (1999). *On General Secondary Education* (Law of Ukraine). Retrieved from zakon.rada.gov.ua/laws/show/651-14 (in Ukr.).

[2]Cabinet of Ministers of Ukraine (2010). *About Approval of Regulations about Comprehensive Educational Institutions* (Resolution). Retrieved from zakon.rada.gov.ua/laws/show/778-2010-%D0%BF (in Ukr.).

[3]Boaler, J. (2000). *Multiple Perspectives of Mathematics Teaching and Learning (International Perspectives on Mathematics Education).* Westport: Praeger.

[4]Pruski, L. & Friedman, J. (2014). An Integrative Approach to Teaching Mathematics, Computer Science, and Physics with Matlab. *Mathematics and Computer Education, 48 (1).*

[5]Ward-Penny, R. (2011). *Cross-Curricular Teaching and Learning in the Secondary School... Mathematics* (1st ed.). London: Routledge.

[6]Dodd, H. (1991). An Integrated Approach to Teaching Mathematics and Science. *Teaching Mathematics Applications, 10 (4),* 138.

[7]Johnston, J., Ní, R. & Gráinne, M. (2014). An integrated approach to the teaching and learning of science and mathematics utilising technology: the teachers' perspective. *i-manager's Journal on School Educational Technology, 9 (4),*16.

cycles[8 9 10 11] (Pruski & Friedman, 2014; Dodd, 1991; Johnston & Grainee, 2014; Lee & Hollebrands, 2008; ect.), as well as with the subjects of social and humanities cycle[12 13 14] (Boaler, 2000; Ward-Penny, 2011; Mueller, Yankelewitz & Maher, 2011; etc.). Summarizing the opinions of the above mentioned researchers, it is appropriate to note that integrative relations of mathematics define its role in the educational process in different ways. Integration with related subjects has applied nature, it is aimed at achieving practical purpose, and mathematics for these subjects is knowledge and methodological resource which promotes formation of subject-related skills of the pupils. The realization of integrative relations of mathematics with subjects of social and humanities cycle have educational and developmental perspective, takes place in a logical and linguistic, communicative, artistic, and aesthetic aspects, promotes the formation of general skills (e.g., perceptual, communicative, etc.) and personal qualities.

The aim of the research is to generalize foreign experience of the realization of integrative connections of mathematics in the extra-class activities in comprehensive educational institutions, to identify features, forms and methods of extra-class activities in mathematics and other subjects.

An important aspect of sustainable educational development is changes in the modern society which have a significant impact on education in general and mathematics education in particular. A fundamental component of the modernization of educational content around the world is extending of borders and new opportunities facing modern youth seeking to obtain the educational skills in various fields of today.

The modern European scientists implicitly emphasize that knowledge and skills of young people should be based on improving their competence[15] (Johnston, Ní, & Gráinne, 2014). With a view to introducing provisions on acquiring a certain level of competence in the field of defining learning outcomes in mathematics the curricula were reviewed and changes thereto have been introduced in many European countries.

In the collection of works *Mathematics Education in Europe: Common Challenges and National Policies*[16] (2011) compiled by the Education, Audiovisual and Culture Executive Agency (EACEA) there are examples of the organization of extra-curricular activities in mathematics in different European countries. It is noted that the educational policies of most of these countries defined the integrated nature of extra-class activities in mathematics, which is held during the recess periods, after classes, at weekends and holidays. There is a noteworthy fact that the purpose of extra-class activities in Europe is the identification and development of talented pupils. However, in the UK extra-class work is aimed at increasing the motivation of pupils with different natural abilities to study mathematics. The program STEM (Science, Technology, Engineering, and

---

[8] Pruski, L. & Friedman, J. (2014). An Integrative Approach to Teaching Mathematics, Computer Science, and Physics with Matlab. *Mathematics and Computer Education, 48 (1).*

[9] Dodd, H. (1991). An Integrated Approach to Teaching Mathematics and Science. *Teaching Mathematics Applications, 10 (4),* 138.

[10] Johnston, J., Ní, R. & Gráinne, M. (2014). An integrated approach to the teaching and learning of science and mathematics utilising technology: the teachers' perspective. *i-manager's Journal on School Educational Technology, 9 (4),*16.

[11] Lee, H., & Hollebrands, K. (2008). Preparing to Teach Mathematics with Technology: An Integrated Approach to Developing Technological Pedagogical Content Knowledge. *Contemporary Issues in Technology and Teacher Education* [Online serial], *8(4).*

[12] Boaler, J. (2000). *Multiple Perspectives of Mathematics Teaching and Learning (International Perspectives on Mathematics Education).* Westport: Praeger.

[13] Ward-Penny, R. (2011). *Cross-Curricular Teaching and Learning in the Secondary School Mathematics* (1st ed.). London: Routledge.

[14] Mueller, M., Yankelewitz, D., & Maher, C. (2011). Sense Making as Motivation in Doing Mathematics: Results from Two Studies. *The Mathematics Educator, 20(2),* 33-43.

[15] Johnston, J., Ní, R. & Gráinne, M. (2014). An integrated approach to the teaching and learning of science and mathematics utilising technology: the teachers' perspective. *i-manager's Journal on School Educational Technology, 9 (4),*16.

[16] EACEA. (2011). *Mathematics Education in Europe: Common Challenges and National Policies*. Brussels, Eurydice.102-104.

Mathematics) has been implemented in the country and provides extra-class activities in mathematics through the integration of subjects of natural, technological and physics and mathematics cycles.

In other countries, there is a program MST (Mathematics, Science, and Technology) aimed at ensuring the implementation of integrative relations of mathematics with subjects of natural and technological cycles. As a part of the program, different level competitions are organized for the pupils (local, regional, and national), pupils may also participate in international competitions. For example, the Mathematical Community in the Republic of Cyprus in cooperation with the Ministry of Education organizes local and national competitions at all levels of education and encourages pupils and students to participate in the international events. Similar competitions are held in Germany and France.

In Austria, there is the governmental project IMST (Innovationen machen Schulen Top) aimed at implementing innovation in academic and extra-class activities in mathematics and the subjects of natural and technological cycles.

In Ireland, the main mechanism of teaching mathematics is basing the instructional material on specific examples for the students to get the information on the subject and develop the ability to solve problems.

In Poland they use key recommendation of the programs thus setting the link between mathematics and daily life[17] (Parveva, Noorani, Ranguelov, Motiejunaite, & Kerpanova, 2012).

Finland established an institutional framework to improve the educational process in mathematics and related subjects. There is LUMA Centre in the country, which aims at implementing the MST program at all levels and ensuring cooperation between comprehensive schools, universities and businesses. The activity of the Centre is coordinated by the Faculty of Natural Sciences of the University of Helsinki. With the Centre's assistance, mathematical summer schools, camps to combine education and recreation are created for gifted pupils, and training courses and workshops for teachers are conducted. It is reasonable to note that LUMA is also a resource centre that provides educational institutions with necessary methodical support.

In Spain, the presentation of the material in mathematics is based on the use of didactic elements which are well known to students. The use of such elements in higher educational institutions and specialized schools is aimed at implementing a specific task in such a way that will enable each of the students to solve the task, thus preparing them for adult life. The program includes such activities as reflection, planning arrangements and their implementation; creating hypotheses or variability of solving[18] (Parveva, Noorani, Ranguelov, Motiejunaite & Kerpanova, 2012).

In Estonia, some schools organize summer courses for pupils who have achieved the best results in mathematics and related subjects.

In Belgium, active learning is also preferable. It is treated as an important element of the development of confidence, autonomy and creativity of students. A specific direction in the presentation of the material is focused work with students, selection of a particular time for reflection and realization of the task set that contributes to the acquisition of critical attitude and encourages a more systematic and flexible thinking which is the foundation of deep study of the basics of mathematics and a more detailed approach to the analysis[19] (Parveva, Noorani, Ranguelov, Motiejunaite & Kerpanova, 2012).

In Liechtenstein, comprehensive schools hold special events every year, such as Einstein Week. These events aim at peer-learning and activity-based learning, which

[17] Parveva, T., Noorani, S., Ranguelov, S., Motiejunaite, A. & Kerpanova, V. (2012). *Mathematics Education in Europe: Common Challenges and National Policies.* Warsaw: FRSE, 178. (In Polish).

[18] Parveva, T., Noorani, S., Ranguelov, S., Motiejunaite, A. & Kerpanova, V. (2012). *Mathematics Education in Europe: Common Challenges and National Policies.* Warsaw: FRSE, 178. (In Polish).

[19] Parveva, T., Noorani, S., Ranguelov, S., Motiejunaite, A. & Kerpanova, V. (2012). *Mathematics Education in Europe: Common Challenges and National Policies.* Warsaw: FRSE, 178. (In Polish).

provides for the application of knowledge in everyday life through preparation and implementation of various projects by pupils.

In Slovenia, one of the priorities is the link of physical culture and mathematical foundations of knowledge. The model of the development of physical and motor abilities at the same time with cognitive ones is part of teaching mathematics. The aim is to collect numerical data of the physical principles of training and work with them in the classroom, their grouping and discussion[13].

In Italy, extra-class activities in mathematics and related subjects for senior school pupils are research-oriented according to the program *Scientific Degrees* funded by the Ministry of Education. The program aims at attracting pupils to the school scientific communities and doing research; cooperation between comprehensive schools and higher educational institutions. In addition, under the project Promotion of Excellence various contests and competitions are held for gifted pupils.

The analysis of the studies of foreign authors[20 21 22 23] (Boaler, 2000; Pruski & Friedman, 2014; Ward-Penny, 2011; Schuepbach, 2015; etc.) makes it possible to identify the following features of the implementation of integrative relations of mathematics in the process of extra-class activities:

- development of pupils' interest in knowledge, positive attitudes towards learning mathematics and other subjects in aggregate, understanding the relation between them with the aim of a comprehensive vision of problems to be addressed;
- obtaining integrated knowledge that rely on relation, synthesis of knowledge on different subjects in order to form pupils' integrated system of view of the world, the development of integrative thinking;
- formation of integrated skills for complex (diverse) problem solving, the use of methods of one science to study objects of other sciences.

The next question to be addressed is the use of integrated methods during extra-class activities in mathematics and other subjects. The researchers[24 25] (Stillman, Kaiser, Blum & Brown, 2013; Ward-Penny, 2011; etc.) substantiated the expediency of performing integrated projects by the pupils. The purpose of the latter is solving interdisciplinary problems, a comprehensive understanding of knowledge through their use in unusual situations. As R. Ward-Penny notes, while working on these projects pupils put forward and prove hypotheses, create and present models, choose ways of their practical application, develop their own methods of calculating certain phenomena, carry out statistical data processing, make interdisciplinary portfolio.

It is reasonable to distinguish the following stages of the arrangement of the integrated project:

1. *Preliminary*. At this stage it is important to separate disciplines on which the project will be implemented; to determine the volume of mathematical material, the nature of subjects' integration (unilateral or bilateral); statement of problem, goals and objectives.
2. *Planning* stage involves identifying the sources of information; selection of methods of information collection and analysis, the ways of presenting results of the

---

[20]Boaler, J. (2000). *Multiple Perspectives on Mathematics Teaching and Learning (International Perspectives on Mathematics Education).* Westport: Praeger.

[21]Pruski, L. & Friedman, J. (2014). An Integrative Approach to Teaching Mathematics, Computer Science, and Physics with Matlab. *Mathematics and Computer Education, 48 (1).*

[22]Ward-Penny, R. (2011). *Cross-Curricular Teaching and Learning in the Secondary School... Mathematics* (1st ed.). London: Routledge.

[23]Schuepbach, M. (2015). Effects of Extracurricular Activities and Their Quality on Primary School-Age Students' Achievement in Mathematics in Switzerland. *School Effectiveness and School Improvement: An International Journal of Research, Policy and Practice, 26 (2),* 279-295.

[24]Stillman, G. A., Kaiser, G., Blum, W. & Brown, J. P. (Eds.). (2013). *Teaching Mathematical Modelling: Connecting to Research and Practice.* Netherlands: Springer, 127.

[25]Ward-Penny, R. (2011). *Cross-Curricular Teaching and Learning in the Secondary School... Mathematics* (1st ed.). London: Routledge.

final product; establishing procedures and criteria for assessment of the project; distribution of tasks among members of the search team.

3. At the *research* stage the problem should be solved, and to do so the pupils conduct a search from diverse sources. At this stage, generalization and discussion of the findings are also made. A number of authors led by G. Stillman[26] (2013) emphasize that discussing information by the members of the search group, putting forward and justification of different (often contradictory) perspectives, making collective decisions reinforce associative links and therefore promotes deeper learning.

4. The stage of *presentation* of the project results. The members of search team report on the progress, present the results of the project, for example in the form of charts, diagrams, figures, pivot tables, etc.

5. The stage of *assessment and summarizing*. The project results should be assessed by a teacher of mathematics, teachers of school subjects, audience (pupils, parents) and the project authors themselves. As R. Ward-Penny[27] (2011) rightly observes the criteria for assessment should be developed beforehand and on their basis the assessment sheets should be prepared.

We would like to mention mind-mapping as an integrated method which is appropriate to be applied in the extra-class activities in mathematics and other subjects. The basis of mind-mapping is associative approach and provisions of modern science that human actions on a conscious or unconscious level are defined by concepts or ideas in the brain.

The developer of mind-mapping T. Buzan[28] (2013) emphasizes that a mind map is a chain of associations that come from the central concept or head to it. Creating a mind map allows visual structuring of the material, adding new information, understanding the problem from different perspectives. The structure of the map corresponds to the natural logics of the brain, clearly reflecting the mental activity of pupils, it activates thinking and strengthens memory.

The scientists[29] (Mueller, Yankelewitz & Maher, 2011; etc.) believe that the use of mind maps contributes to the realization of integrative relations of mathematics with other subjects, for example during the hobby groups' and the pupils' scientific communities' sessions. The pupils are informed about the problem which is to be addressed on the basis of prior knowledge. A pupil runs into the material unfamiliar to him and using imagination, associative approach creates a certain image, picture, diagram, drawing, plan, etc. The central concept (keyword) is located in the middle of the sheet, and then a pupil draws branches which relate to the basic concepts. To differentiate the main and the secondary idea, the main idea can be drawn in one colour, and secondary – in some others. It is reasonable to show the relations between the main and secondary points with arrows and lines. On the mind map pupils can always chart extra points that arise in the process of thinking or discussion with others.

It is interesting that mathematical methods are applied not only in the process of the integration of mathematics with related subjects, but also subjects of social and humanitarian cycles. Thus, one of such methods is modeling, that is the research of certain phenomena, processes and objects through building their models. According to the researchers[30] (Stillman, Kaiser, Blum & Brown, 2013), modeling is a special kind of character-sign idealization. This method enables the transition from formal

---

[26] Stillman, G. A., Kaiser, G., Blum, W. & Brown, J. P. (Eds.). (2013). *Teaching Mathematical Modelling: Connecting to Research and Practice.* Netherlands: Springer, 127.

[27] Ward-Penny, R. (2011). *Cross-Curricular Teaching and Learning in the Secondary School Mathematics* (1st ed.). London: Routledge.

[28] Buzan, T. (2013). *Modern Mind Mapping for Smarter Thinking.* Cardiff Bay: Proactive Press.

[29] Mueller, M., Yankelewitz, D., & Maher, C. (2011). Sense Making as Motivation in Doing Mathematics: Results from Two Studies. *The Mathematics Educator, 20(2),* 33-43.

[30] Stillman, G. A., Kaiser, G., Blum, W. & Brown, J. P. (Eds.). (2013). *Teaching Mathematical Modelling: Connecting to Research and Practice.* Netherlands: Springer, 127.

mathematical task to its interpretation and “illustration”; enhances mental activity of pupils, improves the efficiency of learning, the formation of scientific thinking, awareness of the unity of theory and practice. Scientists point out that mathematical modeling scheme consists of the following components:

1. Singling out certain essential features of the phenomenon, process, object; rejecting minor details; replacement of semantic concepts with formal mathematical equivalents; model design.
2. Logical analysis of the model, calculation, obtaining results.
3. Obtaining new information about the initial object.
4. Interpretation, model clarification.

According to J. B. Muciaccia[31] (2011), the method of modeling can be used when learning a foreign language, including acquiring foreign-language knowledge and skills in extra-class activities, for example, during meetings of the English language club. The material to be learnt is divided into the following steps to create an integrated system of knowledge:

1. Presentation of information (listening + visualization).
2. Perception of information by the pupils who are offered to use techniques proposed of eidetic memory – a system of memorizing material using figurative memory (“drawing” images of words in mind), collective discussion of created models.
3. Re-presentation of the material and understanding of information through logical thinking, checking correspondence of the meanings of words to created images.
4. Practical application of the material in speech. Pupils are offered to come up with their own creative examples of words and phrases, make stories, poems, etc.

We may conclude that in European countries mathematics extra-class activities are integrated and conducted in a juncture with other subjects. The main purpose of extra-class activities is to find gifted pupils and help them develop, increasing the motivation to study mathematics and subjects of natural, technological, social, and humanities cycle. The forms of extra-class activities are the organization of competitions, contests, mathematics summer schools, courses and camps; holding subject weeks, meetings of hobby groups, organizing scientific communities. The integrated methods to be used in extra-class activities in mathematics and other subjects are the implementation of integrated projects, creation of mind maps, and use of the modeling method. The prospects for further research are the study of integrative relationships of mathematics with other school subjects in terms of rural schools based on the analysis of domestic and foreign experience.

[31] Muciaccia, J. B. (2011). *Thinking in English: A New Perspective on Teaching ESL.* Lanham: R&L Education.

REFERENCES

A Nordic Recognition Network (NORRIC) Country Report. (2009). The Educational System of Ukraine. Retrieved from http://norric.org/files/education-systems/Ukraine2009.

Abramov, A. M. (2003). *On the status of mathematics education in high school (1978-2003).* Moscow, Russia: Fazis. (In Rus.).
Абрамов А. М. О положении с математическим образованием в средней школе (1978—2003). – М.: Фазис, 2003. –130 с.

Akulenko, I.A. (2013). *Theoretical and Methodical Principles of Developing Methodical Competence of Prospective Mathematics Teacher of Specialized School.* (Unpublished Doctoral Dissertation). Cherkasy Bohdan Khmelnytsky National University, Cherkasy, Ukraine. (In Ukr.).
Акуленко І. А. Теоретико-методичні засади формування методичної компетентності майбутнього вчителя математики профільної школи : дис. ... д-ра пед. наук : спец. 13.00.02 «Теорія і методика навчання (математика)» / Ірина Анатоліївна Акуленко ; Черкас. нац. ун-т імені Богдана Хмельницького. – Черкаси, 2013. – 688 с.

Akulenko, I.A., Klyatska, L.M. & Ous, I.V. (2007). *Algebra and Number Theory: Teaching methodical guide for organizing classroom and independent work. Part I. The theory of divisibility in the ring of integers.* Cherkasy, Ukraine: Ed. from. Bohdan Khmelnytsky National University of Cherkasy. (In Ukr.).
Акуленко І. А. Алгебра і теорія чисел : навч.-метод. посібник для організації аудиторної та самостійної роботи студентів. Ч. І. Теорія подільності в кільці цілих чисел / Акуленко І. А., Кляцька Л. М., Ус І. В. – Черкаси : Вид. від. ЧНУ імені Богдана Хмельницького, 2007. – 72 с.

Alpers, B. (2013). A Framework for Mathematics Curricula in Engineering Education. Retrieved from http://www.sefi.be/wpcontent/uploads/Competency%20based% %20curriculum%20incl%20ads. pdf.

Althauz, D. & Dum, E. (1984). *Color, form, amount. Experience in the development of cognitive abilities of the preschool children* (V. V. Yurtaykin Trans. from germ.). Moscow, Russia: Prosveshchenie (In Rus.).
Альтхауз Д. Цвет, форма, количество. Опыт работы по развитию познавательных способностей детей дошкольного возраста / Пер. с нем.; под ред. В. В. Юртайкина / Д. Альтхауз, Е. Дум. – М. : Просвещение, 1984. – 64 с.

Anderson, J. R. (1983). *The architecture of cognition. (Cognitive science series, N 5).* Cambridge (Mass.), London: Harvard Univ. Press. 345 p.

Andreiev, V. I. (1981). *Heuristic programming of educational and research activities*. Moscow, Russia: Vyisshaya shkola. (In Rus.).
Андреев В. И. Эвристическое программирование учебно-исследовательской деятельности / В. И. Андреев. – М.: Высш. шк., 1981. – 240 с.

Aplers, B. (2014). Competence-oriented assignments in the mathematical education of mechanocal engineers. Proceedings of SEFI 42nd annual conference. Birmingham, UK. Retrieved from http://www.sefi.be/conference-2014/0012.pdf.

Appleby, J., Samuels, P., Treasure-Jones, T. (1997). Diagnosis – A knowledge-based diagnostic test of basic mathematical skills. *Computers and Education*, *28* (2), 113-131.

Arnold, V. I. (1997). Mathematics and mathematical education in the modern world. *Mathematical Education*, *2*, 22–23. (In Rus.).
Арнольд В. И. Математика и математическое образование в современном мире. / В. И. Арнольд // Математическое образование. – 1997. – № 2. – С. 22–23.

Artemova, L. V. (1997). *Color, form, size, and number.* Kyiv, Ukraine: Tomiris (In Ukr.).
Артемова Л. В. Колір, форма, величина, число / Л. В. Артемова. – К. : Томіріс, 1997. – 176 с.

Avramenko, M. (2007). *Profile Education in General School in the Federal Republic of Germany* (PhD thesis). The Institute of Pedagogy of APS Academy of Pedagogical Science of Ukraine, Kyiv, Ukraine. (In Ukr.).
Авраменко М. М. Профільне навчання в середніх школах Федеративної Республіки Німеччини: автореф. дис. на здобуття наук. ступеня канд. пед. наук: спец. 13.00.01 / М. М. Авраменко. – К., 2007. – 20 с.

Babanskij, YU. K. (1985). *Methods of teaching in a modern comprehensive schoo*l. Moscow, Russia: Prosveshchenie. (In Rus.).
Бабанский Ю. К. Методы обучения в современной общеобразовательной школе / Ю. К. Бабанский. – М. : Просвещение, 1985. – 208 с.

Baglaeva, N. (1999). *Current approaches to the logical-mathematical development of preschool children*. *Pre-school education, 7*, 3-4 (In Ukr.).
Баглаєва Н. Сучасні підходи до логіко-математичного розвитку дошкільнят / Н. В. Баглаєва. // Дошкільне виховання. – 1999. – №7. – С. 3–4.

Baglaeva, N. (2002). *Logical-mathematical development of preschoolers: ways of optimization*. *Teacher's palette, 2*, 12-14 (In Ukr.).
Баглаєва Н. Логіко-математичний розвиток дошкільників: шляхи оптимізації / Баглаєва Н. // Палітра педагога. – 2002. – № 2. – С. 12–14.

Baglaeva, N. I. (1997). *Individually differentiated approach to the development of the mathematical notions of the six year olds* (Unpublished Doctoral dissertation). Institute of Pedagogy APS of Ukraine, Kiev, Ukraine. (In Ukr.).
Баглаєва Н. І. Індивідуально-диференційований підхід до формування математичних уявлень у дітей шостого року життя : дис. … канд. пед. наук: спец. 13.00.01 / Н. І. Баглаєва. – Київ, 1997. – 174 с.

Baryaeva, L. B. (2005). *Integrative model of mathematical education of preschoolers with time-lagged psychical development* (Unpublished Doctoral dissertation). Moscow State University of Education, Moscow, Russian. (In Rus.).
Баряева Л. Б. Интегративная модель математического образования дошкольников с задержкой психического развития : дис. ... докт. пед. наук: спец. 13.00.03 / Л. Б. Баряева. – Москва, Московский государственный открытый педагогический университет им. М. А. Шолохова 2005. – 405 с.

Beloshistaya, A. V. (2008). *Development of mathematical capabilities of preschoolers: theoretical and practical issues.* Moscow, Russian: Publishing house of the Moscow psycholodical-social institute; Voronezh, Russian: NPO «MODEK». (In Rus.).
Белошистая А. В. Развитие математических способностей дошкольников: вопросы теории и практики / А. В. Белошистая. – М. : Издательство Московского психолого-социального института ; Воронеж : НПО «МОДЭК», 2008. – 352 с.

Berezivska, L., Bibik, N. & Burda, M. et al. (2003). The concept of subject-specialized education in high school (Approved by committee decision of the Ministry of Education and Science of Ukraine on 25.09.03 № 10 / 12-2). Academy of Pedagogical Sciences of Ukraine. The Institute of pedagogy. *Inform. Coll. Of Ministry of Education and Science of Ukraine, 24*, 3-15. (In Ukr.).
Концепція профільного навчання в старшій школі [Затв. рішенням колегії М-ва освіти і науки України від 25.09.03 № 10/12-2] / АПН України. Ін-т педагогіки; Уклад.: Л. Березівська, Н. Бібік, М. Бурда та ін. // Інформ. зб. М-ва освіти і науки України. – 2003. – № 24. – С. 3-15.

Bershadskiy, M. E. (2004). *Understanding as a pedagogical category.* Moscow, Russian: Pedagogicheskiy poisk (In Rus.).
Бершадский М. Е. Понимание как педагогическая категория. (Мониторинг когнитивной сферы: Понимает ли ученик, что изучает?) / М. Е. Бершадский. – М. : Центр «Педагогический поиск», 2004. – 176 с.

Bevz, G. P. (1995). *Mathematics. Grade 11*. Kiev, Ukraine: Osvita. (In Ukr.).
Бевз Г. П. Математика. 11 клас / Г. П. Бевз. – К. : Освіта, 1995. – 192 с.

Bibik, N. (2008). Competence in teaching. *Encyclopedia of education* (Academy of Pedagogy of Ukraine, Ed. by Kremen', V. G.). Kyiv, Ukraine: Yurinkom Inter. (In Ukr.).
Бібік Н. М. Компетентність у навчанні / Н. М. Бібік // Енциклопедія освіти; Акад. пед. наук України ; голов. ред. В. Г. Кремінь. – К. : Юрінком Інтер, 2008. – С. 408-409.

Bibik, N. M. (1997). *Developing Cognitive Interest of Younger Students.* Kyiv, Ukraine: IZMN. (In Ukr.).
Бібік Н. М. Формування пізнавальних інтересів молодших школярів : навч. метод. посіб. для вчителів поч. кл. / Н. М. Бібік. – К. : ІЗМН, 1997. – 92 с.

Bila, I. M. (2011). *Theoretical analysis of the problems of understanding*. *Studies and education of the talented child: theory and practice.* Kyiv, Ukraine: Informatsiyni sistemi, issue 5, 71-82 (In Ukr.).
Біла І. М. Теоретичний аналіз проблеми розуміння / І. М. Біла // Навчання і виховання обдарованої дитини: теорія і практика: зб. наук. праць. – Випуск 5. – К. : Інформаційні системи, 2011. – С. 71-82.

Boaler, J. (2000). *Multiple Perspectives of Mathematics Teaching and Learning* (International Perspectives on Mathematics Education). Westport: Praeger.

Boesen, J., Lithner, J., Palm, T. (2010). The relation between types of assessment tasks and the mathematical reasoning students use. *Educational Studies in Mathematics*, *75* (1), 89-105.

Bogatyreva, I. M. (2008). About enhancing the developmental function of tasks in mathematical course in the 5th and 6th grades. *Mathematics in school*, *6*, 27-32. (In Ukr.).
Богатирьова І. М. Про посилення розвивальної функції задач у курсі математики 5–6 класів / І. М. Богатирьова // Математика в школі. – 2008. – № 6. – С. 27–32.

Bogoyavlenskaya, D. (1983). *Intellectual activity as a problem of creativity. Monograph.* Rostov-na-Donu, Russia: Publishing house of Rostov University. (In Rus.).
Богоявленская Д. Б. Интеллектуальная активность как проблема творчества : монография / Д. Б. Богоявленская. – Ростов-на-Дону : Издательство Ростовского университета, 1983. – 176 с.

Boltyanskii, V. G. & Grudenov, Y. I. (1988). How to learn to look for solutions of problems. – *Mathematics in school, 1*, 8-14. (In Rus.).
Болтянский В. Г. Как учить поиску решения задач / В. Г. Болтянский, Я. И. Груденов // Математика в школе. – 1988. – № 1. – С. 8-14.

Bondarevskyi, V. B. (1969). *Interest in Knowledge and in Self-Education*. *Developing Students' Striving to Self-Education.* Volgograd, Russia: VGPI. 51-69. (In Rus.).
Бондаревский В. Б. Интерес к знаниям и проблеме самообразования / В. Б. Бондаревский // Формирование у учащихся стремления к самообразованию. – Волгоград : ВГПИ, 1969. – С. 51–69.

Bosovsky, M.V. (2005). The theory of borders, for the implementation of continuity. *Bulletin of Cherkasy University, Series: Pedagogical Science, 74*, 3-8. (In Ukr.).
Босовський М. В. Теорія границь, напрямки здійснення наступності / М. В. Босовський // Вісник Черкаського університету. Серія Педагогічні науки. – Вип. 74. – Черкаси: Вид. від. ЧНУ ім. Б. Хмельницького, 2005. – С. 3-8.

Boykina, D. V. & Milloushev, V. B. (1995). Geometry problemss that are solved by using a circle. *Scientific Papers of PU "Paisii Hilendarski"*, *32* (2), 145-152. (In Bul.]
Бойкина Д. В. Задачи по геометрия, които се решават с помощна окръжност / Д. В. Бойкина, В. Б. Милушев // Научни трудове на ПУ „Паисий Хиилендарски". – 1995. – №32 (2). – С. 145-152.

Bozovich, L. I. (1966). *Age Laws of Forming Child's Personality* (Abstract of Doctoral Dissertation). Moscow, Russia. (In Rus.).
Божович Л. И. Возрастные закономерности формирования личности ребенка : автореф. дис. … д-ра психол. наук / Л. И. Божович . – М., 1966. – 29 с.

Breen, R., Lindsay, R., Jenkins A., & Smith, P. (2001). The Role of Information and Communication Technologies in a University Learning Environment. *Studies in Higher Education*, *26* (1), 95-114.

Brezhneva, E. G. (1997). *Developing cognitive activity of the children of the senior preschool age (based on mathematics).* (Unpublished Doctoral dissertation). Institute of Pedagogy APS of Ukraine, Kiev, Ukraine. (In Rus.).
Брежнева Е. Г. Формирование познавательной активности у детей старшего дошкольного возраста (на материале математики) : дис. … канд. пед. наук: спец. 13.00.01 / Е. Г. Брежнева. – Киев, Институт педагогики АПН Украины, 1997. – 168 с.

Burda, M. I., & Tarasenkova, N.A. (2007). *Geometry. Textbook for the 7th grade of the secondary school.* Kyiv, Ukraine: Publishing House "Osvita". (In Ukr.).
Бурда М. І. Геометрія : [підруч. для 7 кл. загальноосвіт. навч. закладів] / М. І. Бурда, Н. А. Тарасенкова. – К. : Зодіак-ЕКО, 2007. – 208 с.

Burda, M. I., & Tarasenkova, N. A. (2008). *Geometry. Textbook for the 8th grade of the secondary school.* Kyiv, Ukraine: Publishing House "Osvita". (In Ukr.).
Бурда М. І. Геометрія : [підруч. для 8 кл. загальноосвіт. навч. закладів] / М. І. Бурда, Н. А. Тарасенкова. – К. : Зодіак-ЕКО, 2008. – 240 с.

Burda, M. I., & Tarasenkova, N. A. (2009). *Geometry. Textbook for the 9th grade of the secondary school.* Kyiv, Ukraine: Publishing House "Osvita". (In Ukr.).
Бурда М. І. Геометрія : [підруч. для 9 кл. загальноосвіт. навч. закладів] / М. І. Бурда, Н. А. Тарасенкова. – К. : Зодіак-ЕКО, 2009. – 240 с.

Burda, M. I., & Tarasenkova, N. A. (2010). *Geometry. Textbook for the 10th grade of the secondary school.* Kyiv, Ukraine: Publishing House "Osvita". (In Ukr.).
Бурда М. І. Геометрія : [підруч. для 10 кл. загальноосвіт. навч. закладів; академічний рівень] / М. І. Бурда, Н. А. Тарасенкова. – К. : Зодіак-ЕКО, 2010. – 176 с.

Burda, M. I., Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M., & Serdiuk, Z. O. (2013). *Geometry. Textbook for the 11th grade of the secondary school.* Kyiv, Ukraine: Publishing House "Osvita". (In Ukr.).
Бурда М. І. Геометрія : [підруч. для 11 кл. загальноосвіт. навч. закладів; академічний та профільний рівні] / М. І. Бурда, Н. А. Тарасенкова, І. М. Богатирьова, О. М. Коломієць, З. О. Сердюк. – К. : Видавничий дім "Освіта", 2013. – 304 с.

Buzan, T. (2013). *Modern Mind Mapping for Smarter Thinking.* Cardiff Bay: Proactive Press.

Byelich, N. I. (2012). *Involvement of students in research work*. Retrieved from http://osvita.ua/school/lessons_summary/upbring/27192/. (In Ukr.).
Бєліч Н. І. Залучення учнів до науково-дослідницької роботи / Н. І. Бєліч. [Електронний ресурс]. – Режим доступу: http://osvita.ua/school/lessons_ _summary/upbring/27192/. [Доступ: 02 липня, 2016]

Carr, M., Bowe, B., Ni Fhloinn, E. (2013). Core skill assessment to improve mathematical competency. *European Journal of Engineering Education*. *38*(6). 608-619.

Carr, M., Fidalgo, C., Bigotte de Almeida, M.E. (2015). Mathematics diagnostic testing in engineering: an international comparison between Ireland and Portugal. *European Journal of Engineering Education. 40*(5). 546-556. doi: 10.1080/03043797.2014.967182

Cerņajeva, S., Volodko, I. (2012). Teaching Mathematics in Riga Technical University. *International Conference "Mathematical Competence Development in Higher Education Institutions"*, Latvia, Jelgava, 14 December, 2012, 21.

Cerņajeva, S., Volodko, I. (2016). Improvement of Teaching Methodology of Mathematics for Students and Pupils Using the MOOC Platform. *15th International Scientific Conference "Engineering for Rural Development": Proceedings. Vol.15*, Latvia, Jelgava, 25-27 May, 2016. Jelgava, 1286-1290.

Cettel R. B. (1971). *Abilities: the structure grows and action.* Boston: Noughtoun Mifflim. (In Eng.).

Chashechnykova, O. S. (1997). *The development of mathematical abilities of the secondary school pupils* (Unpublished Doctoral dissertation). Kiev, Ukraine. (In Ukr.).
Чашечникова О. С. Розвиток математичних здібностей учнів основної школи. – Дисс. … канд. пед. н., спеціальність 13.00.02 / Чашечникова О. С. – К., 1997. – 208 с.

Cherkaska, L.P., Matias, O.I., & Marchenko, V.A. (2014). *The problem of the Development of the pupils' cognitive interest.* Poster session presented at the International Distance Methodical Conference "ITM * plus – 2014", Sumy, Sumy State University, p.p. 43-45. (In Ukr.).
Черкаська Л. П. До проблеми формування пізнавального інтересу учнів / Л. П. Черкаська, Л. О. Матяш, В. О. Марченко // Розвиток інтелектуальних умінь і творчих здібностей учнів та студентів у процесі навчання дисциплін природничо-математичного циклу «ITM*плюс – 2014» : матеріали Міжнар. дистанційної наук.-метод. конф. (20–21 березня 2014 р., м. Суми) : у 3-х ч. / упоряд. О. С. Чашечникова. – Суми : СумДУ, 2014. – Ч. 1. – С. 43–45.

Cherkasov, R. S. (1976). About methodical preparation of math teacher in pedagogical institute. *Mathematics in school, 5,* 80-84. (In Rus.).
Черкасов Р. С. О методической подготовке учителя математики в педагогическом вузе / Р. С. Черкасов // Математика в школе. – 1976. – № 5. – С. 80-84.

Cherkasov, R. S. (1997). The history of national school math education. *Mathematics in school, 4, 5, 6* (In Rus.).
Черкасов Р. С. История отечественного школьного математического образования / Р. С. Черкасов // Математика в школе. – 1997. – №4, 5, 6.

Chernets'ka, T. I. (2011). *Modern lesson: theory and practice of modeling*. Kyiv, Ukraine: TOV "Praymdruk". (In Ukr.).
Чернецька Т. І. Сучасний урок: теорія і практика моделювання : [навч. посібник] / Т. І. Чернецька. – К. : ТОВ «Праймдрук», 2011. – 352 с.

Conception of the target social program of upgrading school natural and mathematical education for the period till 2015. № 1720 (2010, 27 August). Retrieved from http://zakon.golovbukh.ua/regulations/8451/467747/ (In Ukr.).
Концепція цільової соціальної програми підвищення якості природничо-математичної освіти на період до 2015 року від 7 серпня 2010 р. № 1720. [Електронний ресурс]. Режим доступу: http://zakon.golovbukh.ua/regulations/8451/4677

Crampton, A., Vanniasinkam, T., Milic, N. (2010). Vodcasts! How to unsuccessfully implement a new online tool. *Interaction in Communication Technologies and Virtual Learning Environments: Human Factors*, 118-128.

Culter, G.L., Pulko, S.H. (2002). Investigation UK undergraduate electrical and electronic engineering attrition. *International Journal of Electrical Engineering Education. 39*(3). 181-191.

D'Alembert, J. (1950). *Dynamics*. Moscow-Leningrad, Russia: Gostekhizdat. (In Rus.).
Даламбер Ж. Л. Динамика / Ж. Л. Даламбер; пер. з франц. В. П. Егоршина. – Москва-Ленинград: Гостехиздат, 1950. – 344 с.

Danilov, M. A., & Esipov, B. P. (1957). *Didaktika.* Moscow, Russian: Uchpedgiz. (In Rus.).
Данилов М. А. Дидактика / М. А. Данилов, Б. П. Есипов. – М. : Учпедгиз, 1957. – 518 с.

Davydov, V. V. (1995). On the concept of developmental education. *Pedagogy*, *1*, 29-32. (In Rus.).
Давыдов В. В. О понятии развивающего обучения / В. В. Давыдов // Педагогика. – 1995. – № 1. – С. 29-32.

Davydov, V. V. & Andronov, V. V. (1997). Psychological terms of origin of ideal actions. Mathematical abilities of preschoolers. *Psychological science and education*, *3*, 27–41 (In Rus.).
Давыдов В. В. Психологические условия происхождения идеальных действий. Математические умения у дошкольников / В. В.Давыдов, В. П. Андронов // Психологическая наука и образование. – 1997. – №3. – С. 27–41.

Demidova, J. O. (2007). *Developing the fundamental principles of cognitive independence of the senior preschoolers through the constructional activity.* (Abstract of Doctoral dissertation). NAPS Institute of Problems of Education of Ukraine, Kiev, Ukraine. (In Ukr.).
Демидова Ю. О. Формування першооснов пізнавальної самостійності старших дошкільників у конструктивній діяльності : автореф. дис. …канд. пед. наук: спец. 13.00.08 / Ю. О. Демидо-ва. – Київ, Інститут проблем виховання НАПН України, 2007. – 20 с.

Diltey, V. (2001). *Hermeneutics and theory of literature.* In V. Bibihina & N. Plotnikova (Eds.). Collected works in six volumes (Vol. 1-6). Moscow, Russian: Dom intellektualnoy knigi (In Rus.).
Дильтей В. Собрание сочинений в 6 тт. Т.4: Герменевтика и теория литературы / Под ред. А. В. Михайлова и Н. С. Плотникова / Пер. с нем. под ред. В. В. Бибихина и Н. С. Плотникова. – Научное издание. – М. : Дом интеллектуальной книги, 2001. – 400 с.

Doblaev, L. P. (1974). *Developing the techniques of understanding the text. Text and cognitive activity.* Saratov, Russian: Publishing house of Saratov univercity (In Rus.).
Доблаев Л. П. Формирование приемов понимания текста / Л. П. Доблаев // Текст и познавательная деятельность. Саратов : Изд-во Сарат. ун-та, 1974. – С. 50–122.

Dobrov, D. (2012). *Teaching of Mathematics in school and Kolmogorov's reform*. Retrieved from http://www.dm-dobrov.ru/history/mathematics.html
Добров Д. Преподавание математики в школе и реформа А. Н. Колмогорова : [Електронний ресурс]. Режим доступу: http://www.dm-dobrov.ru/history/mathematics.html

Dodd, H. (1991). An Integrated Approach to Teaching Mathematics and Science. *Teaching Mathematics Applications, 10* (4), 137-143. doi: 10.1093/teamat/10.4.137

Druz, B. G. (1978). *Training Cognitive Interests of Younger Students in Education Process*. Kyiv, Ukraine: Radianska Shkola. (In Ukr.).
Друзь Б. Г. Виховання пізнавальних інтересів молодших школярів у процесі навчання / Б. Г. Друзь. – К. : Радянська школа, 1978. – 128 с.

Druz, Z.V. (1997). *Non-standard Tasks as a Means of Stimulating Cognitive Interests of Senior Students*. (Abstract of Doctoral Dissertation). Taras Shevchenko National University, Kyiv, Ukraine. (In Ukr.).
Друзь З. В. Нестандартні завдання як засіб стимулювання пізнавальних інтересів молодших школярів : автореф. дис. … канд. пед. наук : спец. 13.00.01 «Загальна педагогіка та історія педагогіки» / Зінаїда Василівна Друзь ; Криворізький держ. педагогічний ін-т. ; Кривий Ріг, 1997. – 24 с.

Dunlosky, J., Rawson, K.A., Marsh, E.J., Nathan, M.J., Willingham, D.T. (2013). Improving students' learning with effective learning techniques: Promising directions from cognitive and educational psychology. *Psychological Science in the Public Interest, Supplement*, *14* (1), 4-58.

Dybova, M. V. & Maslova, S. V. (2011). Aim and pithy aspect of the concept of "Competence task". *Bulletin of the Volga University named after V. N. Tatishcheva, 8*. Retrieved from http://cyberleninka.ru/article/n/tselevoy-i-soderzhatelnyy-aspekt-ponyatiya-kompetentnostnaya-zadacha (In Rus.).
Дубова М. В. Целевой и содержательный аспект понятия «компетентностная задача» / М. В. Дубова, С. В. Маслова // Вестник Волжского университета им. В.Н. Татищева. – 2011. – № 8 : [Электронный ресурс] : Режим доступа : http://cyberleninka.ru/article/n/tselevoy-i-soderzhatelnyy-aspekt-ponyatiya-kompetentnostnaya-zadacha

EACEA. (2011). *Mathematics Education in Europe: Common Challenges and National Policies*. Brussels: Eurydice. doi:10.2797/72660

Egulemova, N.N. (2003). *Modification of Geometric Problems as a Means of Developing Cognitive Interest of Basic School Students* (Abstract of the PHD Dissertation). Ministry of Education of the Russian Federation, Orel, Russia. (In Rus.).
Егулемова Н. Н. Видоизменение геометрических задач как средство развития познавательного интереса учащихся основной школы : автореф. … канд. пед. наук : спец. 13.00.02 «Теория и методика обучения и воспитания (математика)» / Наталья Николаевна Егулемова ; Мин-во образования Российской Федерации. – Орел, 2003. – 18 с.

Elkonin, D. B. (1989). *Selected psychological works.* Retrieved from http://psychlib.ru/mgppu/eit/EIT-001-.HTM#Раздел_I . (In Rus.).
Эльконин Д. Б. Избранные психологические труды / Д. Б. Эльконин. – М. : Педагогика, 1989. – 560 с. – С. 36–45. – [Електронный ресурс]. Режим доступа: http://psychlib.ru/mgppu/eit/EIT-001-.HTM#Раздел_I

Elkonin, D. B. (1971). On the problem of periodization of mental development in children. *Questions of psychology, 4*, 6-20. (In Rus.).
Эльконин Д. Б. К проблеме периодизации психического развития в детском возросте / Д. Б. Эльконин // Вопросы психологии. – 1971. – № 4. – С. 6-20.

Erdniev, P. M. & Mikerova G. G. (2004). *Principles of demonstrastion, systematic character and sequence in the technology of large-sized didactic units at school.* In E. Malinochka (Ed.). Krasnodar, Russia: Kubanskiy State University (In Rus.).
Эрдниев П. М. Принципы наглядности, систематичности и последовательности в технологии укрупненных дидактических единиц школе / П. М. Эрдниев, Г. Г. Микерова; под ред. Э. Г. Малиночка. – Краснодар : Кубанский гос. ун-т, 2004. – 177 с. – С.12–24.

Fabre, A. (1978). *Scientific pedagogy and education.* Parise, France: Libr. Colin. (In Fr.).
Fabre A. Pédagogie scientinque et education / A. Farbe. – Parise: Libr. Colin, 1978.

Fanninger, L. (2008). *Peculiarities of profile-oriented education in the secondary schools in Austria* (Unpublished Doctoral dissertation). Ternopol National Pedagogical University named after W. Gnatjuk, Ternopol, Ukraine. (In Ukr.).
Фаннінгер Л. П. Особливості профільного навчання в основній школі Австрії : дис. ... канд. пед. наук : 13.00.01 / Фаннінгер Людмила Павлівна. – Тернопіль, 2008. – 186 с.

Faulkner, F., Hannigan, A., Gill, O. (2010). Trends in the mathematical competency of university entrants in Ireland by leaving certificate mathematics grade. *Teaching Mathematics and its Application. 29*(2), 76-93. doi:10.1093/teamat/hrq002

Fidler, M. (1981). *Mathematics in kindergarten.* Moscow, Russia: Prosveshchenie (In Rus.)
Фидлер М. Математика уже в детском саду / М. Фидлер. – М., 1981. – 184 с.

Frenkev, D. G., Milloushev, V. B. & Boykina, D. V. (2007). A complex model of the process of solving mathematical problems of a certain type. *Mathematics and Mathematical Education.* 429-435. Sofia, Bulgaria: BAS. (In Bul.).
Френкев Д. Г. Комплексен модел на процеса решаване на математически задачи от определен вид / Д. Г. Френкев, В. Б. Милушев, Д. В. Бойкина // Математика и математическо образование. – София: Изд. на БАН, 2007. – С. 429-435.

Fridman, L. M. & Turetskii, E. N. (1984). *How to learn to solve problems*. Moscow, Russia: Prosveshcheniye. (In Rus.).
Фридман Л. М. Как научиться решать задачи / Л. М. Фридман, Е. Н. Турецкий. – Москва: Просвещение, 1984.

Fridman, L. M. (1998). *Theoretical basis of the methodology of education in mathematics.* Moscow, Russia: Flinta. (In Rus.).
Фридман Л. М. Теоретические основы методики обучения математике / Л. М. Фридман. – Москва: Флинта, 1998.

Fridman, L. M. (1987). *Teaching experience through the eyes of a psychologist*. Moscow, Russia: Prosveshchenie. (In Rus.).
Фридман Л. М. Педагогический опыт глазами психолога / Л. М. Фридман. – М.: Просвещение, 1987. – 223 с.

Funtikova, O. A. (1992). *Use of models in developing the preschoolers (of 5-7 years of age) knowledge about time.* (Unpublished Doctoral dissertation). Institute of Pedagogy APS of Ukraine, Kiev, Ukraine (In Rus.).
Фунтикова О. А. Использование моделей в формировании знаний о времени у дошкольников 5-7 лет : дис. … канд. пед. наук: спец. 13.00.01 / О. А. Фунтікова. – Київ, Институт педагогики АПН Украины, 1992. – 205 с.

Furtak, B. & Zhyvko D. (2000). New approaches to the content of mathematics education in Ukraine. *Mathematics in school, 5,* 24-30 (In Ukr.).
Фуртак Б. Нові підходи до змісту математичної освіти в Україні / Б. Фуртак, Д. Живко // Математика в школі. – 2000. – №5. – С. 24–30.

Gadamer, H. - G. (1988). *Truth and method*. Moscow, Russia: «Progress», 1988 (In Rus.).
Гадамер Х. - Г. Истина и метод / Х. - Г. Гадамер. – М. : «Прогресс», 1988. – 704 с.
Galbraith, P., Haines, C. (1998). Disentangling the nexus: Attitudes to mathematics and technology in a computer learning environment. *Educational Studies in Mathematics*, *36* (3), 275-290.
Galperin, P. Y. & Georgiev, L. S. (1994). *Developing the initial mathematical concepts*. In Z. A. Mihaylova & R. L. Nepomnyashchaya (Eds.), *Theory and methods of developing mathematical concepts of preschoolers: reading-book in 6 parts.* Part 3, 312 p. Moscow, Russia: St. Petersburg (In Rus.).
Гальперин П. Я. Формирование начальных математических понятий / П. Я. Гальперин, Л. С. Георгиев // Теория и методика развития математических представлений у дошкольников: хрестоматия в 6 частях. – М. : СПб., 1994. – Ч. 3. – 312 с.
Ganchev, I. (1988). Logical models in the methodology of education in mathematics. In: *Mathematics and informatics*, Part II, (Proceedings of the "Jubilee Conference on the occasion of the 25th anniversary of the University of Shumen"), (pp. 49-52). Shumen, Bulgaria. (In Bul.).
Ганчев И. Логически модели в методиката на обучението по математика / И. Ганчев // Математика и информатика : Сборник доклади „Юбилейна научна конференция 25 години Шуменски университет". – Част II. – Шумен, 1988. – С. 49-53.
Ganchev, I. (1999). *Basic learning activities in mathematics class* (*synthesis of the results of different studies*). Sofia, Bulgaria: Modul-96. (In Bul.).
Ганчев И. Основни учебни дейности в урока по математика (синтез на резултати от различни изследвания) / И. Ганчев. – София: Модул-96, 1999.
Ganchev, I. (2001). Analysis and synthesis in mathematics education. In: V. Miloushev (Ed.), *Methods for solving problems* (*from the school course in mathematics*). Part I, 5-31. Plovdiv, Bulgaria: Makros. (In Bul.).
Ганчев И. Анализът и синтезът в обучението по математика / И. Ганчев // Методи за решаване на задачи (от училищния курс по математика); Научен ред. В. Милушев. – Ч. I. –Пловдив, България: Макрос, 2001. – С. 5-31.
Ganchev, I. & Kuchinov, Y. (1996). *Organization and methodology of the lesson in mathematics*. Sofia, Bulgaria: Modul. (In Bul.).
Ганчев И. Организация и методика на урока по математика / И. Ганчев, Й. Кучинов. – София: Модул, 1996.
Gardner, H. (1999). *Intelligence Reframed: Multiple Intelligences for the 21st Century.* Basic Books, Psychology. 292 p. Pp.33-34.
Garifullina, I. V. (2003). *Pedagogical Environment for Developing Cognitive Interests of Senior Schoolchildren in Interaction with Nature* (Abstract of the PHD Dissertation). Surgut State University, Surgut, Russia. (In Rus.).
Гарифуллина И. В. Педагогические условия развития познавательного интереса младших школьников во взаимодействии с природой : автореф. дисс. … канд. пед. наук : спец. 13.0.01 «Общая педагогика, история педагогики и образования» / Ирина Витальевна Гарифуллина ; Сургутский государственный университет. – Сургут, 2003. – 16 с.

Gaydarzhiyskaya, L. P. (1996). *Developing the principles of mathematical notions in the children of the senior preschool age.* (Unpublished Doctoral dissertation). Institute of Pedagogy APS of Ukraine, Kiev, Ukraine. (In Rus.)
Гайдаржийская Л. П. Формирование элементов математических представлений у детей старшего дошкольного возраста : дис. … канд. пед. наук: спец. 13.00.01 / Л. П. Гайдаржийская. – Киев, Институт педагогики АПН Украины, 1996. – 173 с.

Geidzs, N. L., Berliners, D. C. (1999). *Pedagogical psychology*. Riga, Latvia: Zvaigzne ABC. (In Latv.)
Geidžs, N. L., Berliners, D. C. Pedagoģiskā psiholoģija / N. L. Geidžs, D. C. Berliners. – Rīga: Zvaigzne ABC, 1999. – 662.lpp.

Georgieva, M. (2001). *Reflection in mathematics education* (*V-VI class*). V. Turnovo, Bulgaria: Feber. (In Bul.).
Георгиева М. Рефлексията в обучението по математика (V-VI клас) / М. Георгиева. – В. Търново: Фебер, 2001.

Georgieva, M., & Grozdev, S. (2015). *Morfodinamics of the development of noospheric intelligence*. Sofia, Bulgaria. (In Bul.).
Георгиева М. Морфодинамиката за развитието на ноосферния интеллект / М. Георгиева, С. Гроздев. – София, 2015.

Gershynskyy, B. S. (2002). *Philosophy of education in XXth century*. Moscow, Russia: Pedagogicheskoe obshchestvo Rossii. (In Rus.).
Гершунский Б. С. Философия образования для XXI века. / Б. С. Гершунский. – М. : Педагогическое общество России, 2002. – 512 с.

Gill, O., O'Donoghue, J., Faulkner, F., Hannigan, A. (2010). Trends in performance of science and technology students (1997–2008) in Ireland. *International Journal of Mathematical Education in Science and Technology. 41*(3). 323-339. doi: 10.1080/00207390903477426.

Glushkov, V.M., Chernikov, S.N. (2010). Review about scientific and pedagogical activity of head of the department of geometry and mathematics teaching methods Cherkasy Pedagogical Institute, candidate of physical and mathematical sciences, associate professor Semenovich A.F.: (Preprint, 1970]. In N. Tarasenkova (Ed.) *International scientific and methodical conference "Problems of mathematical education" (PME – 2010), Cherkasy, 24-26 November 2010* (pp. 17-19). Cherkasy, Ukraine (In Rus.).
Глушков В. М., Черников С. Н. Отзыв о научной и педагогической деятельности заведующего кафедрой геометрии и методики преподавания математики Черкасского педагогического института, кандидата физико-математических наук, доцента Семеновича А. Ф. : (препринт] // Матеріали міжнародної науково-методичної конференції "Проблеми математичної освіти" (ПМО – 2010), м. Черкаси, 24-26 листопада 2010 р. / Н. А. Тарасенкова (відп. ред.). – Черкаси: Вид. від. ЧНУ ім. Б. Хмельницького, 2010. – С. 17-19.

Gnedenko, B. V. (2006). *On teaching mathematics at universities and pedagogical institutions at the turn of the trird millenium*. Moscow, Russia: ComKniga. (In Rus.).
Гнеденко Б. В. Об обучении математике в университетах и педвузах на рубеже двух тысячелетий. / Б. В. Гнеденко, Д. Б. Гнеденко. – М. : КомКнига, 2006. – 160 с.

Goold, E., Devitt, F. (2012). The role of mathematics in engineering practice and in the formation of engineers. In Avdelas A. (Ed.) *Proceedings of SEFI 40th annual conference "Engineering Education 2020: Meet the Future"*. Thessaloniki, Greece.

Gordiєnko, T. (1998). *Profile differentiation in teaching physics in the 10- 11 forms of a f secondary school (humanities profile)* (Abstract of the PhD thesis). Institute of Pedagogy of APS Academy of Pedagogical Science of Ukraine, Kyiv, Ukraine. (In Ukr.).
Гордієнко Т. П. Профільна диференціація навчання фізики в 10-11 класах середньої загальноосвітньої школи (гуманітарний профіль): автореферат дисертації на здобуття наукового. ступеня канд. пед. наук: спец. 13.00.02 / Т. П. Гордієнко. – К., 1998. – 21 с.

Grebennikova, O. A. (2005). *Project Activities as a Means of Developing Cognitive Interests of Senior Students.* (Abstract of the PHD Dissertation). Novgorod Yaroslav the Wise State University, Veliky Novgorod, Russia. (In Rus.).
Гребенникова О. А. Проектная деятельность как средство развития познавательных интересов старшеклассников : автореф. дисс. … канд. пед. наук : спец. 13.00.01 «Общая педагогика, история педагогики и образования» / Оксана Александровна Гребенникова; Новгородский государственный университет им. Ярослава Мудрого. – Великий Новгород, 2005. – 22 с.

Grin, R. & Lakson, V. (1982). *Introduction to the world of numbers.* Moscow, Russia: Pedagogika (In Rus.).
Грин Р. Введение в мир числа / Р. Грин, В. Лаксон. – М.: Педагогика, 1982. – 193 с.

Grozdev, S. (2002). Organization and self-organization in solving problems. *Mathematics and informatics*, *6*, 51-58. (In Bul.)
Гроздев С. Организация и самоорганизация при решаване на задачи / С. Гроздев // Математика и информатика. – 2002. – №6. – С. 51-58.

Grozdev, S. (2007). *For high achievements in mathematics.The Bulgarian Experience* (*Theory and Practice*). Sofia, Bulgaria: Kota.

Grozdev, S. & Kenderov, P. (2005) Instruments of recognizing and supporting gifted students in mathematics. In: *Mathematics and Mathematical Education,* (pp. 53-63). Sofia, Bulgaria: BAS. (In Bul.).
Гроздев С. Инструментариум за откриване и подкрепа на изявени ученици по математика / С. Гроздев, П. Кендеров // Математика и математическо образование”. – София: Изд. на БАН, 2005. – С. 53-63.

Guzeiev, V. V. (2002). *Methods and organizational forms of learning*. Moscow, Russia: Narodnoe obrazovanie. (In Rus.).
Гузеев В. В. Методы и организационные формы обучения / В. В. Гузеев. – М.: Народное образование, 2002. – 128 с.

Hegel, Georg. *The Science of Logic*. Translated by W. H. Johnston and L. G. Struthers. London: George Allen & Unwin, 1929.

Holodnaya, M. A. (2002). *The Psychology of Intellect.* St. Petersburg, Russia: Piter. (In Rus.).
Холодная М. А. Психология интеллекта / М. А. Холодная. – СПб: Питер, 2002. – 272 с.

Honcharova, I.V. (2006). Methods of Developing Student’s Personality in Mathematics Heuristic Electives. *Bulletin of the Cherkasy University. Series: Pedagogical Sciences, 93,* 30-35. (In Ukr.).
Гончарова І. В. Прийоми розвитку особистості учня на евристичних факультативах з математики / І. В. Гончарова // Вісник Черкаського університету. Серія Педагогічні науки. – Черкаси : вид. від. ЧНУ ім. Б. Хмельницького, 2006. – Вип. 93. – С. 30–35.

Hubachov, A.I. (2009). *Didactic Conditions of the Individualization of Pre-Profile Training of High School Students.* (Unpublished PHD dissetation). Institute of Pedagogy of Ukraine APS, Kyiv, Ukraine. (In Ukr.).
Губачов О. І. Дидактичні умови індивідуалізації допрофільної підготовки учнів основної школи : дис. ... канд. пед. наук : спец. 13.00.09 «Теорія навчання» / Олександр Ілліч Губачов; Ін-т педагогіки АПН України. – К., 2009. – 251 с.

Ignatenko, M. J. (1997). *Activation of high school students' learning activities in studying mathematics*. Kyiv, Ukraine: Tyrash. (In Ukr.).
Ігнатенко М. Я. Активізація навчально-пізнавальної діяльності учнів старших класів при вивченні математики / М. Я. Ігнатенко. – К. : «Тираж», 1997. – 300 с.

Johnston, J., Ní, R. & Gráinne, M. (2014). An Integrated Approach to the Teaching and Learning of Science and Mathematics Utilising Technology: the Teachers' Perspective. *I-Manager's Journal on School Educational Technology, 9* (4),14-26.

Kagan, M. S. (1974). *Human activities: Experience of System Analysis*. Moscow, Russian: Politizdat. (In Rus.).
Каган М. С. Человеческая деятельность: Опыт системного анализа / М. С. Каган. – М.: Политиздат, 1974. – 328 с.

Kalmykova, Z. I. (1981). *Productive thinking as basis of learnability.* Moscow, Russia: Pedagogika (In Rus.).
Калмыкова З. И. Продуктивное мышление как основа обучаемости / З. И. Калмыкова. – М. : Педагогика, 1981. – 200 с.

Kent, P., Noss, R. (2003). *Mathematics in the University Education of Engineers. A Report to The Ove Arup Foundation*. Retrieved from http://www.lkl.ac.uk/ /research/REMIT/Kent-Noss-report-Engineering-Maths.pdf.

Kharlamov, I. F. (1997). *Pedagogy.* Moscow, Russia: Yurist. (In Rus.).
Харламов И. Ф. Педагогика / И. Ф. Харламов. – М.: Юрист, 1997. – 512 с.

Khinchin, O.Ya. (2013). *Education papers: Issues of teaching mathematics. Fighting methodical clichés*. (Ed. B.V. Gnedenko). Moscow, Russia: ComKniga. (In Rus.).
Хинчин А. Я. Педагогические статьи: Вопросы преподавания математики. Борьба с методическими штампами / А. Я Хинчин. Под ред. и с предисл. Б. В. Гнеденко. – М.: КомКнига, 2013. – 208 с.

Kintch, W. & van Dijk, T. A. (1978). Toward a model of text comprehension and production. *Psychological review*, v. 85. pp. 363-394.

Kiseleva, T. (2006). The problem of periodization in the studies on the history of mathematical education. *Bulletin of Eletski State University named after I.A. Bunin. A series of "History and theory of mathematics education", 11,* 132-140 (In Rus.).
Киселева Т. В. Проблема периодизации в исследованиях по истории математического образования / Т. В. Киселева // Вестник Елецкого государственного университета им. И.А.Бунина. Вып. 11. Серия «История и теория математического образования». – Елец: ЕГУ им. И.А.Бунина, 2006. – С. 132-140.

Kislenko, K, Grevholm, B., & Lepik, M. (2007). Mathematics is important but boring: students' beliefs and attitudes towards mathematics. In C. Bergsten, B. Masoval, & F. Ronning (Eds.), *Relating practice and research in mathematics education. Proceedings of Norma05, Fourth Nordic Conference on Mathematics Education, Trondheim, 2nd-6th of September 2005*. Trondheim, Norway: Tapir Akademisk Forlag. 349–360.

Kniazian, M. O. (2007). Formation of self-research as a pedagogical problem in the theory and practice of preparing future teachers. *Problems of Education, 51*, 92-98. (In Ukr.).
Князян М. О. Формування самостійно-дослідницької діяльності як педагогічна проблема в теорії і практиці підготовки майбутнього вчителя / М. О. Князян // Проблеми освіти. – Київ, 2007. – Вип. 51. – С. 92-98.

Kogalovskii, S.R. (2006). On the leading plans of teaching mathematics. *Pedagogy, 1,* 39-48 (In Rus.).
Когаловский С. Р. О ведущих планах обучения математике / С. Р. Когаловский // Педагогика. – 2006. – №1. – С. 39–48.

Kolmogorov, A. N. (1988). *Mathematics – science and profession.* (Ed. G. Galperin). Moscow, Russia: Nauka. Glavnaiya redaktsiya fiziko-matematicheskoy literatury. (In Rus.).
Колмогоров А. Н. Математика – наука и профессия. / А. Н. Колмогоров. Сост. Г. А. Гальперин. – М. : Наука. Главная редакция физико-математической литературы, 1988. – 288 с.

Kolmogorov, A. M., Abramov, O. M., Dudnitsyn, Y. P., Ivlev, B. M. & Shvartsburd, S. I. (1992). *Algebra and Introduction into Analysis. 10-11 grades.* Kyiv, Ukraine: Osvita. (In Ukr.).
Колмогоров А. М. Алгебра і початки аналізу. 10-11 клас / А. М. Колмогоров, О. М. Абрамов, Ю. П. Дудніцин та ін.] ; за ред. А. М. Колмогорова. – К. : Освіта, 1992. – 350 с.

Kolmogorov, A. N., Semenovich, A. F., & Cherkasov, R. S. (1979). *Geometry: Text-book for 6-8 grades of the secondary school*. Moscow, Russia: Prosveshcheniye. (In Rus.).
Колмогоров А. Н. Геометрия : учеб. пособие для 6-8 кл. средней школы / А. Н. Колмогоров, А. Ф. Семенович, Р. С. Черкасов; под ред. А. Н. Колмогорова. – М.: Просвещение, 1979. – 382 с.

Kolyagin, Y. M. & Oganesyan, V. A. (1980). *Learn to solve problems*. Moscow, Russia: Prosveshcheniye. (In Rus.).
Колягин Ю. М. *Учись решать задачи* / Ю. М. Колягин, В. А. Оганесян. – Москва: Просвещение, 1980.

Kolyagin, Y.M. (2001). *Russian school and mathematics education: Our pride is our pain*. Moscow, Russia: Prosveshcheniye (In Rus.).
Колягин Ю. М. Русская школа и математическое образование: Наша гордость наша боль. – М.: Просвещение, 2001. – 318 с.

Kondratieva, G. V. (2009). On the problem of the periodization of the development of school mathematics education in Russia. *Bulletin of Moscow State Regional University. "Pedagogy" series, 3,* 124-131 (In Rus.).
Кондратьева Г. В. К вопросу о периодизации развития школьного математического образования в России // Вестник Московского государственного областного университета. Серия «Педагогика». – № 3. – М.: Изд-во МГОУ, 2009. – С. 124-131.

Korobov, E. (2005). Understanding as a didactic problem. *Moscow psychological magazine*, *11*. Retrieved from http://magazine.mospsy.ru/nomer11/s10.shtml.
Коробов Е. Понимание как дидактическая проблема [Електронный ресурс] / Е. Коробов // Московский психологический журнал. – 2005. – № 11. – Режим доступа: http://magazine.mospsy.ru/nomer11/s10.shtml.

Kostenko, I. P. (2012). Reform of school mathematics in 1970-1978 years. On the 40th anniversary of the «Kolmogorov reform». *«Alma mater» (Bulletin of the Higher School)*. Retrieved from https://almavest.ru/ru/node/1256 (In Rus.)
Костенко И. П. Реформа школьной математики 1970-1978 гг. К 40-летию «Колмогоровской реформы» //«Alma mater» (Вестник высшей школы) : [Электронный ресурс]. Режим доступа: https://almavest.ru/ru/node/1256

Kostenko, I. P. (2014). 1965 – 1970 years. Organizational preparation of the reform-70 ME, APS, personnel, programs, textbooks (Article Five). *Math education, 3(71),* 2–18 (In Rus.).
Костенко И. П. 1965 – 1970 гг. Организационная подготовка реформы-70: МП, АПН, кадры, программы, учебники (статья пятая) / И. П. Костенко // Матем. обр. – № 3(71). – 2014. – С. 2–18.

Kostyuk, G. S. (1989). *Educational process and psychical development of personality.* In L. M. Prokolienko (Ed.). Kyiv, Ukraine: Radyanska shkola (In Ukr.).
Костюк Г. С. Навчально-виховний процес і психічний розвиток особистості / Г. С. Костюк. – К. : Радянська школа, 1989. – 608 с. – С. 255; С.300–307.

Kostyuk, G. S. (1950). On the psychology of understanding. *Scientific papers of the SRI of psychology, 2*, 7-57. (In Ukr.).
Костюк Г. С. Про психологію розуміння / Г. С. Костюк // Наукові записки НДІ психології. – К. : 1950. – Т. 2. – С. 7–57.

Kovalenko, V. G., Krivosheev, V. Y., & Lemberskyy, L. Y. (1991). *Algebra. 8th grade: experimental textbook for schools with advanced study of mathematics and special schools of Physics and Mathematics profile.* Kyiv, Ukraine: Osvita. (In Ukr.).
Коваленко В. Г. Алгебра. 8 клас : експериментальний навчальний посібник для шкіл з поглибленим вивченням математики і спеціальних шкіл фізико-математичного профілю / В. Г. Коваленко, В. Я. Кривошеєв, Л. Я. Лемберський. – К. : Освіта, 1991. – 302 с.

Kovalenko, V. G., Krivosheev, V. Y., & Staroseltseva, O. V. (1992). *Algebra. 9th grade: experimental textbook for schools with advanced study of mathematics and special schools of Physics and Mathematics profile.* Kyiv, Ukraine: Osvita. (In Ukr.).
Коваленко В. Г. Алгебра. 9 клас : експериментальний навчальний посібник для шкіл з поглибленим вивченням математики і спеціалізованих шкіл фізико-математичного профілю / В. Г. Коваленко, В. Я. Кривошеєв, О. В. Старосельцева. – К. : Освіта, 1992 . – 271 с.

Kozlova, V. A. (2003). *Developing the elementary mathematical notions among the junior children.* (Unpublished Doctoral dissertation). Moscow State University of Education, Moscow, Russian. (In Rus.).
Козлова В. А. Формирование элементарных математических представлений у детей младшего возраста : дис. … докт. пед. наук: спец. 13.00.02 / В. А. Козлова. – М. : Московский государственный открытый педагогический университет им. М. А. Шолохова, 2003. – 308 с.

Kremen', V. (Ed.). (2008). *Encyclopedia of education* (Academy of Pedagogy of Ukraine). Kyiv, Ukraine: Yurinkom Inter. (In Ukr.).
Енциклопедія освіти / Акад. пед. наук України ; голов. ред. В. Г. Кремінь. – К. : Юрінком Інтер, 2008. – 1040 с.

Kremen', V. G., & Il'in, V. V. (2005). *Philosophy: thinkers, ideas and concepts. Textbook.* Kyiv, Ukraine: Kniga. (In Ukr.).
Кремень В. Г., Ільїн В. В. Філософія: мислителі, ідеї, концепції: Підручник / В. Г. Кремень, В. В. Ільїн. – К.: Книга. – 2005. – Т. 528.

Krupich, V. (1995). *Theoretical basis of teaching how to solve school mathematical problems.* Moscow, Russia: Prometei. (In Rus.).
Крупич В. Теоретические основы обучения решению школьных математических задач / В. Крупич. – Москва: Прометей, 1995.

Krutetskiy, V. (1972). *Essentials of pedagogical psychology*. Moscow, Russia: Prosvieshcheniie. (In Rus.).
Крутецкий В. А. Основы педагогической психологии / В. А. Крутецкий. – М. : Просвещение, 1972. – 255 с.

Kuhar, A.V. (1984). *Developing Students' Cognitive Interest in Learning Mathematics in the 4-7th Forms.* (Unpublished PHD dissertation). Kyiv State Pedagogical University, Kyiv, Ukraine. (In Rus.).
Кухар А. В. Формирование познавательного интереса у учащихся к математике в процессе её изучения в 4–7 классах : дис… канд. пед. наук : спец. 13.00.02 «Методика преподавания математики» / Алевтина Васильевна Кухар; Киевский государственный педагогический университет. – К., 1984. – 191 с.

Kulchytska, G. V. (1999). Cognitive Interests of Senior Students. *Practical Psychology and Social Work*, *6*, 21-23. (In Ukr.).
Кульчицька Г. В. Пізнавальні інтереси старшокласників / Г. В. Кульчицька // Практична психологія та соціальна робота. – 1999. – № 6. – С. 21–23.

Kuzminsky, A. I. (2011). *University Pedagogy*. Kyiv, Ukraine: Znannya. (In Ukr.).
Кузьмінський А. І. Педагогіка вищої школи: навч. посіб. / А. І. Кузьмінський. – К. : Знання,. – 486 с.

Kuzminskyy, A. I., Tarasenkova, N. A., & Akulenko, I. A. (2009). *Scholarly premises for the methodical preparation of the future teachers of mathematics*. Cherkasy, Ukraine: Publishing house of Bogdan Khmelnitsky National University at Cherkasy. (In Ukr.).
Кузьмінський А. І. Наукові засади методичної підготовки майбутнього вчителя математики: монографія / А. І. Кузьмінський, Н. А. Тарасенкова, І. А. Акуленко. – Черкаси : Вид. від. ЧНУ імені Богдана Хмельницького, 2009. – 320 с.

Lalchev, Z. & Vutova, I. (2003). Binding element. *Mathematics and Mathematical Education,* (pp. 369-373). Sofia, Bulgaria: BAS. (In Bul.).
Лалчев З. Свързващ елемент / З. Лалчев, И. Вутова // Математика и математическо образование. – София: Изд. на БАН, 2003. – С. 369-373.

Law of Ukraine "On Education". (In Ukr.). Retrieved from http://zakon4.rada. .gov.ua/laws/show/1060-12.
Закон України «Про освіту» / [Електронний ресурс]. – Режим доступу : http://zakon4.rada.gov.ua/laws/show/1060-12.

Lee, H., & Hollebrands, K. (2008). Preparing to Teach Mathematics with Technology: An Integrated Approach to Developing Technological Pedagogical Content Knowledge. *Contemporary Issues in Technology and Teacher Education [Online serial], 8*(4). Retrieved from http://www.citejournal.org/volume-8/issue-4-08/mathematics/preparing-to-teach-mathematics-with-technology-an-integrated-approach-to-developing-technological-pedagogical-content-knowledge

Leibniz, G. V. (1982). *Works in four volumes*. Moscow, Russia: Mysl.
Лейбниц Г. В. Сочинения в четырех томах / Г.В. Лейбниц. – М.: Мысль, 1982. – Т. І. – 636 с.

Leontiev, A. N. (1975). *Activity, Consciousness, and Personality*. Moscow, Russia: Politizdat. (In Rus.).
Леонтьев А. Н. Деятельность. Сознание. Личность / А. Н. Леонтьев. – М.: Политиздат, 1975. – 304 с.

Lerner, I. Ya. (1970). The construction of the logic of pedagogical research. *Soviet pedagogy, 5*, 3-10. (In Rus.).
Лернер И. Я. О построении логики дидактического исследования / И. Я. Лернер // Советская педагогика. – 1970. – №5. – С. 3-10.

Leushina, A. M. (1974). *Methods of developing the elementary mathematical ideas in the children of the preschool age.* Moscow, Russia: Prosveshchenie (In Rus.).
Леушина А. М. Методика формирования элементарных математических представлений у детей дошкольного возраста / А. М. Леушина. – М. : Просвещение, 1974. – 368 с.

Levchuk, Z. K. & Ermolchik, I. V. (2014). *Theory and methods of teachingf elementary mathematical concepts to the preschool children.* Vitebsk, Belarus: VSU named after P. M. Masherov (In Rus.).
Левчук З. К. Теория и методика формирования элементарных математических представлений у детей дошкольного возраста / З. К. Левчук, И. В. Ермольчик. – Витебск : ВГУ имени П. М. Машерова, 2014. – 48 с.

L'Huilier, S. (1784). *Exposition élémentaire des calculs des principes supérieurs*. Berlin, Germany: Berlin Académie des sciences.

Lodatko, E. O. In S. T. Zolotukhina (Ed.). (2011). *Mathematical culture of the elementary school teacher. Monograph*. Rivne, Ukraine. (In Ukr.).
Лодатко Є. О. Математична культура вчителя початкових класів : монографія / Є. О. Лодатко; ред.: С. Т. Золотухіна; Міжнар. екон.-гуманіт. ун-т ім. С. Дем'янчука. – Рівне, 2011. – 322 с.

Lodatko, E. O. (2011). *Mathematical culture of the primary school teacher*. Rivne-Slovyansk, Ukraine: Entrepreneur B. I. Matorin. (In Ukr.).
Лодатко Є. О. Математична культура вчителя початкових класів. / Є. О. Лодатко. – Рівне-Слов'янськ : Підприємець Маторін Б. І., 2011. – 324 с.

Lokhvytska, L. (2000). *Developing Cognitive Interests of Senior Children in the Learning-through-play Environment*. (Abstract of the PHD dissertation). M. P. Drahomanov National Pedagogical University, Kyiv, Ukraine. (In Ukr.).
Лохвицька Л. В. Формування пізнавальних інтересів дітей старшого віку в навчально-ігровому середовищі : автореф. дис… канд. пед. наук : спец. 13.00.02 «Теорія та методика навчання» / Любов Василівна Лохвицька ; Національний педагогічний ун-т ім. М. П. Драгоманова. – К., 2000. – 21 с.

Lopez, B., Frederick, G., Lent, R., Brown, W., & Steven, D. (1997). Role of social-cognitive expectations in high school students' mathematics-related interest and performance. *Journal of Counseling Psychology. Vol. 44* (1)*, Jan.* 44–52.

Lovyanova, I. (2013). On the Specific Character of Mathematical Education Content Selection in a Subject-Specialised School. *American Journal of Educational Research, 1*(11), 523-527. Retrieved from http://pubs.sciepub.com/education-1-11-11.

Lovyanova, I. V. (2005). *The content of mathematics education and the problem of formation of intellectual skills of high school students*. Cherkasy, Ukraine: Publishing department of Bohdan Khmelnitsky national university of Cherkasy. (In Ukr.).
Лов'янова І. В. Зміст математичної освіти і проблема формування інтелектуальних умінь старшокласників // Матеріали Всеукр. наук.-метод. конференції "Проблеми математичної освіти". – Черкаси: Вид. відділ ЧНУ ім. Б. Хмельницького, 2005. – С. 96-99.

Lovyanova, I. V. (2012). Professional orientation in teaching high school students. *Higher education of Ukraine: theoretical and methodical magazine*, *Special Issue "Pedagogy of high school: methodology, theory and technology", 3* (46), 170-180. (In Ukr.).
Лов'янова І. В. Професійна спрямованість навчання старшокласників // Вища освіта України: Теоретичний та науково-методичний часопис. – № 3 (46). – 2012. – Тематичний випуск «Педагогіка вищої школи: методологія, теорія, технології». – С. 170–180.

Lovyanova, I. V. (2013). On high school students' professional self-determination in the process of learning mathematics in subject-specialized school. *Vestsi BDPU*, *Issue 3*(4)*,* 35-38. (In Rus.).
Ловьянова И. В. О профессиональном самоопределении старшеклассников в процессе обучения математике в профильной школе / И. В. Ловьянова // Весці БДПУ. – Серыя 3. – 2013. – № 4. – С. 35–38.

Lovyanova, I. V. (2013). Retrospective analysis of the problems of teaching mathematics differentiation in comprehensive school. *Science and Education a New Dimension: Pedagogy and Psychology*, *5*, 114-119. Retrieved from http://scaspee.com-/6/post/2013/07/retrospective-analysis-of-the-problem-of-differentiation-of-educating-to-mathematics-at-general-school-i-lovianova.html (In Rus.).

Ловьянова И. В. Ретроспективный анализ проблемы дифференциации обучения математике в общеобразовательной школе / И. В. Ловьянова // Science and Education a New Dimension: Pedagogy and Psychology. – 2013. – Vol. 5. С. 114–119.

LTSN MathsTEAM Project. (2003). *Diagnostic Testing for Mathematics*. Retrieved from http://www.sigma-network.ac.uk/wp-content/uploads/2013/12/ /diagnostic_test.pdf.

Lutchenko, L. I. (2003). *Planning Unaided Learning Activities of the 7-9th Form Students while Teaching Mathematics.* (Unpublished PHD dissertation) M. P. Dragomanov National Pedagogical University, Kyiv, Ukraine. (In Ukr.).
Лутченко Л. І. Організація самостійної навчально-пізнавальної діяльності учнів 7-9 класів при вивченні математики : дис. … канд. пед. наук : 13.00.02 «Теорія та методика навчання (математика)» / Людмила Іванівна Лутченко ; НПУ ім. М. П. Драгоманова. – К., 2003. – 238 с.

Makhmutov, M. I. (1975). *Problem teaching*. Moscow, Russia: Pedagogika. (In Rus.).
Махмутов М. И. Проблемное обучение / М. И. Махмутов. – М.: Педагогика, 1975. – 210 с.

Markova, A., Matis, T., Orlov, A. (1990). *Forming learning motivation*. Moscow, Russia: Prosvieshcheniie. (In Rus.).
Маркова А. К. Формирование мотивации учения / Маркова А. К., Матис Т. А., Орлов А. Б. – М. : Просвещение, 1990. – 192 с.

Mashovets, M. A. & Stetsenko, I. B. (2009). *Why preschhol age kids need mathematics.* Kiev, Ukraine: Shkilniy svit (In Ukr.).
Машовець М. А. Навіщо дошколярику математика / М. А. Машовець, І. Б. Стеценко. – К.: Шкільний світ, 2009. – 125 с.

Mavrova, R. & Millousheva-Boykina, D. (2002). Method of undetermined coefficients. In V. Miloushev (Ed.), *Methods for solving problems* (*from the school course in algebra and analysis*)*, Part II*, 37-46. Plovdiv, Bulgaria: University press "Paisii Hilendarski". (In Bul.)
Маврова Р. Метод на неопределените коефициенти / Р. Маврова, Д. Милушева-Бойкина // Методи за решаване на задачи (от училищния курс по алгебра и анализ); Научен ред. В. Милушев. – Част II. – Пловдив: ПУИ „Паисий Хилендарски", 2002. – С. 37-46.

Melnyk, M.V. (2005). *Developing Professional Knowledge of Vocational School Students in Extracurricular Activity in Science and Mathematical Disciplines.* (Unpublished PHD dissetation). Vinnytsia M. Kotsiubynskyi State Pedagogical University, Vinnytsia, Ukraine. (In Ukr.).
Мельник М. В. Формування професійних знань учнів професійних училищ у процесі позаурочної роботи з природничо-математичних дисциплін : дис. ... канд. наук : спец. 13.00.04 «Теорія та методика професійної освіти» / Микола Васильович Мельник ; Вінницький держ. пед. ун-т імені Михайла Коцюбинського. – Вінниця, 2005. – 186 с.

Merzlyak, A. G., Polonski, V. B. & Yakir, M. S. (1994). *Unexpected steps or one hundred and thirteen lovely problems.* Sofia, Bulgaria: Academic press "Marin Drinov". (In Bul.).
Мерзляк А.Г. Неочаквана стъпка или сто и тринадесет красиви задачи / А. Г. Мерзляк, В. Б. Полонски, М. С. Якир. – София: Акад. изд. „Марин Дринов", 1994.

Mihaylova, Z. A. & Nepomnyashchaya, R. L. (2000). *Theory and methods of developing mathematical notions in preschoolers: reading-book:* [in 4-6 p.]. Moscow, Russia: Ikar (In Rus.).
Теория и методика развития математических представлений у дошкольников: хрестоматия / Сост. З. А. Михайлова, Р. Л. Непомнящая. – М. : Икар, 2000. – 234 с.

Milloushev, V. & Boykina, D. (2002). Graphical method. In: V. Miloushev (Eds.), *Methods for solving problems* (*from the school course in algebra and analysis*). Part II. (pp. 47-65). Plovdiv, Bulgaria: University press „Paisii Hilendarski". (In Bul.).
Милушев В. Графичен метод / В. Милушев, Д. Бойкина // Методи за решаване на задачи (от училищния курс по алгебра и анализ); Научен ред. В. Милушев. – Ч. II. – Пловдив: ПУИ „Паисий Хилендарски", 2002. – С. 47-65.

Milloushev, V. & Frenkev, D. (2004a). Models for solving and creating geometry problems for finding out properties. *Mathematical education: Status and perspectives*. (Proceedings of International Conference), (pp. 22-28), Mogiljov, Belarus: MDU, (A plenary lecture at invitation). (In Rus.)
Милушев, В. & Френкев, Д. Модели для решения и продуцирования геометрических задач на обнаруживание свойств / В. Милушев, Д. Френкев // *Матэматычная адукацыя: сучасны стан і перспектывы*" : Зборнік матэрыялау міжнароднай навуковай канферэнцыі. – Могилев: МГУ, 2004. – С. 22-28. (Пленарен доклад по покана).

Milloushev, V. & Frenkev, D. (2004b). A model for joint implementation of the methods of analysis, synthesis, and parameterization in searching for solutions of geometric problems. In: *Mathematics and Mathematical Education,* (pp. 348-353). Sofia, Bulgaria: BAS. (In Bul.).
Милушев В. Модел на съвместно прилагане на методите анализ, синтез и параметризация при търсене на решения на геометрични задачи / В. Милушев, Д. Френкев // Математика и математическо образование. – София: Изд. на БАН, 2004. – С. 348-353.

Milloushev, V. (2016). Reflexive-synergetic approach in mathematics education. *Strategies for Policy in Science and Education : Bulgarian Educational Journal*, *24* (1), 69-85. (In Bul.).
Милушев В. Рефлексивно-синергетичен подход при обучението по математика / В. Милушев // Стратегии на образователната и научната политика : Научно описане. – 2016. – №24 (1). – С. 69-85.

Milloushev, V. B. & Frenkev, D. G. (2007). The system of activities for mastering generally-logical methods for solving mathematical problems in accordance with the principle of reflection. *Didactics of Mathematics: Problems and Investigations. International Collection of Scientific Works*, *28*, 178-184. Donetsk, Ukraine: DonNU. (In Rus.).
Милушев В. Б. Система деятельностей для овладения общелогическими методами решения математических задач в соответствии с принципом рефлексивности / В. Б. Милушев, Д. Г. Френкев // Дидактика математики: проблемы и исследования : междунар. сб. науч. труд. – Вып. 28. – Донецк: Дон НУ, 2007. – С. 178-184.

Milloushev, V. B. & Frenkev, D. G. (2007b). A system of the variants of transformation mathematical problems – means of intensifying of reflection. In: *"Science, Education and Time as a Care".* (Reports of the Anniversary Scientific Conference with International Participation*,* 30.11.-1.12.2007). (pp. 127-133). Smolyan, Bulgaria. (In Bul.).
Милушев В. Б. Система варианти на преобразуване на математически задачи – средство за активизиране на рефлексия / В. Б. Милушев, Д. Г. Френкев // Науката, образованието и времето като грижа : сб. доклади от Юбилейната научна конференция с международно участие, 30.11.–1.12.2007). – Смолян, 2007. – С. 127-133.

Milloushev, V. B. & Frenkev, D. G. (2008). About one reflexive model of education and its application. *Mathematics and Mathematical Education,* (pp. 61-72). Sofia, Bulgaria: BAS. (In Bul.).

Милушев В. Б. За един рефлексивен модел на обучение и негово приложение / В. Б. Милушев, Д. Г. Френкев // Математика и математическо образование. – София: Изд. на БАН, 2008. – С. 61-72.

Milloushev, V. B. (2002). Method of mathematical induction. In: V. Miloushev (Eds.), *Methods for solving problems* (*from the school course in algebra and analysis*). Part II. (pp. 13-36). Plovdiv, Bulgaria: University press "Paisii Hilendarski". (In Bul.).
Милушев В. Б. Метод на математическата индукция / В. Б. Милушев // Методи за решаване на задачи (от училищния курс по алгебра и анализ); Научен ред. В. Милушев. – Ч. II. –Пловдив: ПУИ „Паисий Хилендарски", 2002. – С. 13-36.

Milloushev, V. B. (2008). *The triad of solving, creating, and transforming mathematical problems activities in the context of the reflexive - synergetic approach.* Abstract of the dissertation to obtain the degree "Doctor of pedagogical sciences" in scientific speciality "Methodology of education in mathematics". Sofia, Bulgaria. (In Bul.).
Милушев В. Б. Триадата дейности решаване, съставяне и преобразуване на математически задачи в контекста на рефлексивно-синергетичния подход : автореф. на дисер. труд … „доктор на педагогическите науки" по научната специалност "Методика на обучението по математика" / В. Б. Милушев. – София, 2008

Milloushev, V. B. (2009a). Reflection and reflective approash in mathematics education. *Bulletin of Cherkasy University, Series: Pedagogical Science*, *143*, 56-69. (In Rus.).
Милушев, В. Б. Рефлексия и рефлексивный подход в обучении математике / В. Б. Милушев // Вісник Черкаського університету. Серія: Педагогічні науки. Вип. 143. Черкаси: Вид. від. ЧНУ ім. Б. Хмельницького, 2009. – С. 56-69.

Milloushev, V. B. (2009b). About the classification of methods for solving mathemdtiks problems. *Scientific Papers of PU „Paisii Hilendarski"*, *46* (2), 77-90. (In Bul.).
Милушев В. Б. Относно класифициране на методите за решаване на математически задачи / В. Б. Милушев // Научни трудове на ПУ „Паисий Хилендарски". – 2009. – Т. 46. – Кн. 2. – С. 77-90.

Milloushev, V., Frenkev, D. & Millousheva-Boikina, D. (2005). Model for teacning in rediscovery of particular methods for mathematical problem solving. *Proceedings of the IIIrd Congress of the Mathematicians of Macedonia*, Struga, 29.IX-2.X.2005, (pp.123-130).

Mirakova, T. N. (2001). *Didactic principles of the humanization of school mathematical education.* (Unpublished Doctoral dissertation). Institute of General Secondary Education of Russian Academy of Education, Moscow, Russia. (In Rus.).
Миракова Т. Н. Дидактические основы гуманитаризации школьного математического образования : дис. … докт. пед. наук: спец. 13.00.02 / Миракова Т. Н. – Москва, Институт общего среднего образования Российской академии образования, 2001. – 465 с.

Morozova, N. G. (1969). *Developing Cognitive Interest of Abnormal Children (In comparison with the norm).* Moscow, Russia: Prosveshchenie. (In Rus.).
Морозова Н. Г. Формирование познавательного интереса у аномальных детей (сравнительно с нормой) / Н. Г. Морозова. – М. : Просвещение, 1969. – 280 с.

Muciaccia, J. B. (2011). *Thinking in English: A New Perspective on Teaching ESL.* Lanham: R&L Education

Mueller, M., Yankelewitz, D., & Maher, C. (2011). Sense Making as Motivation in Doing Mathematics: Results from Two Studies. *The Mathematics Educator, 20*(2), 33-43.

Mykhalin, G. O. (2003). *Professional training of the mathematics teacher while teaching mathematics analysis*. Kyiv, Ukraine: RNNTs "DINIT". (In Ukr.).
Михалін Г. О. Професійна підготовка вчителя математики у процесі навчання математичного аналізу. / Г. О. Михалін. – Київ : РННЦ «ДІНІТ», 2003. – 320 с.

National Doctrine for Development of Education. (2002). *Osvita Ukrainy*, *33.* (In Ukr.).
Національна доктрина розвитку освіти // Освіта України. – 2002. – №33.
National Strategy for Development of Education in Ukraine for the period until 2021, approved by Decree of the President of Ukraine of 25.06.2013 number 344. Retrieved from http://zakon4.rada.gov.ua/laws/show/344/2013 (In Ukr.).
Національна стратегія розвитку освіти в Україні на період до 2021 року, затверджена Указом Президента України від 25.06.2013 № 344 [Электоронний ресурс]. – Режим доступу: http://zakon4.rada.gov.ua/laws/show/344/2013
Newton, I. (1989). *The mathematical principles of natural philosophy*. Moscow, Russia: Nauka. (In Rus.).
Ньютон И. Математические начала натуральной философии. / И. Ньютон. – М.: Наука, 1989. – 711 с.
Nikolsky, S. M. (1983). *Course of mathematical analysis.* Moscow, Russia: Nauka. (In Rus.).
Никольский С. М. Курс математического анализа : [учебник для физ. и мех.-мат. спец. вузов.] : в 2-х т. / С. М. Никольский. – [3-е изд., перераб. и доп.]. – М. : Наука, 1983. – Т. 2. – 448 с.
Niss, M. (2003). Mathematical competencies and the learning of mathematics: The Danish KOM project. In Gagatsis A, Papastravidis S, (Eds.) *Proceedings of 3rd Mediterranean Conference on Mathematics Education*. Athens, Greece.
Obukhov, A. S. (2006). *The development of research activity of pupils*. Moscow, Russia: Prometey, MPGU. (In Rus.).
Обухов А. С. Развитие исследовательской деятельности учащихся / А. С. Обухов. – М.: Прометей, МПГУ, 2006.
Obukhova, L. F., & Burmenskaya, G.V. (Eds.) (2001). *Jan Piaget: Theory. Experiments. Discussions*. Moscow, Russia: Gardariki. (In Rus.).
Жан Пиаже: теория, эксперименты, дискуссия : сб. статей / Сост. и общ. ред. Л.Ф.Обуховой и Г. В. Бурминской; предислов. Л. Ф. Обуховой. – М. : Гардарики, 2001. – 624 с.
On the Approval of Regulations about Comprehensive Educational Institutions. (2010). Resolution of Cabinet of Ministers of Ukraine. Retrieved from zakon.rada.gov.ua/laws/show/778-2010-%D0%BF (In Ukr.).
Постанова Кабінету міністрів України «Про затвердження Положення про загальноосвітній навчальний заклад» [Електронний ресурс]. – 2010. – 27 серп. – Режим доступу: zakon.rada.gov.ua/laws/show/778-2010-%D0%BF
Ortega y Gasset, J. (2010). *Mission of the university*. Moscow, Russia: Publishing House of University "Vysshaya shkola ekonomiki". (In Rus.).
Ортега-и-Гассет Х. Миссия университета. / Х. Ортега-и-Гассет. – М. : Изд. дом Гос. ун-та – Высшей школы экономики, 2010. –144 с.
Ovcharuk, O. (2003). Competencies as a Key to Educational Content Renewal. *Reform Strategy for Education in Ukraine: Education policy recommendations*. Kyiv, Ukraine: KIS. (In Ukr.).
Овчарук О.Л. Компетентності як ключ до оновлення змісту освіти // Стратегія реформування освіти в Україні: рекомендації з освітньої політики. – К.: К.І.С., 2003. – С.13–43.
Palamarchuck, V. F. (1987). *School teaches us to think.* Moscow, Russia: Prosveshchenie. (In Rus.).
Паламарчук В. Ф. Школа учит мыслить / В. Ф. Паламарчук. – М.: Просвещение, 1987. – 208 с.
Papi, F. & Papi, Zh. (1984). *Children and columns.* Moscow, Russia: Pedagogika (In Rus.).
Папи Ф. Дети и графы / Ф. Папи, Ж. Папи. – М. : Педагогика, 1984. – 223 с.

Parveva, T., Noorani, S., Ranguelov, S., Motiejunaite, A. & Kerpanova, V. (2012). *Mathematics Education in Europe: Common Challenges and National Policies*. Warsaw, Poland: FRSE, 178. (In Polish).
Parveva, T. Nauczanie matematyki w Europie: ogolne wyzwania i strategie krajowe / T. Parveva, S. Noorani, S. Ranguelov, A. Motiejunaite, V. Kerpanova. – Warszawa: FRSE, 2012. – 178 s.

Pavlyuk, T. O. (2012). *Teaching the computer-backed counting to the children of the senior preschool age.* (Abstract of Doctoral dissertation). NAPS Institute of Problems of Education of Ukraine, Kiev, Ukraine. (In Ukr.).
Павлюк Т. О. Навчання дітей старшого дошкільного віку лічби з використанням комп'ютера : автореф. дис. … канд. пед. наук: спец. 13.00.08 / Т. О. Павлюк. – Київ, Інститут проблем виховання НАПН України, 2012. – 19 с.

Petrov, M. K. (2004). *History of the European cultural tradition and its issues*. Moscow, Russia: ROSSPEN. (In Rus.).
Петров М. К. История европейской культурной традиции и ее проблемы. / М. К. Петров. – М. : РОССПЭН, 2004. – 776 с.

Petrov, P. D. & Millousheva-Boykina, D. V. (2000). About the skills for solving and formulating mathematics problems. *Scientific Papers of PU „Paisii Hilendarski", 37* (2), 17-23. (In Bul.).
Петров П. Д. Относно уменията за решаване и съставяне на математически задачи / П. Д. Петров, Д. В. Милушева-Бойкина // Научни трудове на ПУ "Паисий Хилендарски". – 2000. –Т. 37, кн. 2. – С. 17-23.

Petrov, P. D. (2003). *Developing the skills for solving the problems from the school course in mathematics*. Stara Zagora, Bulgaria: Kota. (In Bul.).
Петров П. Д. Формиране на умения за решаване на задачи от училищния курс по математика / П. Д. Петров. – Стара Загора: Кота, 2003.

Petrova, V.T. (1999). *Scientific and Methodological Fundamentals of learning mathematical disciplines in universities.* (Unpublished Doctoral dissertation). Moscow, Russia: MHHHU. (In Rus.).
Петрова В. Т. Научно-методические основы интесификации обучения математическим дисциплинам в высших учебных заведениях : дис. докт. пед. наук : 13.00.02 / В. Т. Петрова. – М.: Изд-во МГГГУ, 1999. – 420 с.

Philippou, N. G., & Christou, C. (1998). The Effects of a Preparatory Mathematics Program in Changing Prospective Teachers' Attitudes towards Mathematics. *Educational Studies in Mathematics, 35*, 189-206.

Piaget, J. (2001). *The Psychology of Intelligence.* In D. E. Berlyne translate. London: Routledge.

Pishchalkovska, M. (2007). *Profile education of the school-leavers* (Abstract of the Doctoral dissertation). Institute of Pedagogical Education and Adult Education of the Academy of Pedagogical Sciences of Ukraine, Kyiv, Ukraine. (In Ukr.).
Піщалковська М.К. Система роботи загальноосвітнього навчального закладу з профільного навчання старшокласників: автореф. дис... канд. пед. наук: 13.00.01 / М.К. Піщалковська ; Ін-т пед. освіти і освіти дорослих АПН України. – К., 2007. – 23 с. – укр.

Pletenitska, L. S. & Krutiy, K. L. (2002). *Logically-mathematical development of preschoolers.* Zaporizhzhya, Ukraine: TOV Lips. Ltd (In Ukr.).
Плетеницька Л. С. Логіко-математичний розвиток дошкільників / Л. С. Плетеницька, К. Л. Крутій. – Запоріжжя : ТОВ Ліпс. Лтд, 2002. – 156 с.

Poincaré, H. (1983). *On Science*. Moscow, Russia: Nauka. Chief Editorial Board of Physics and Mathematics Literature. (In Rus.).
Пуанкаре А. О науке. / А. Пуанкаре. – М. : Наука. Главная редакция физико-математической литературы, 1983. – 560 с.

Pometun, O. (2004). Implementation of competence approach is a promising trend of modern education. *Bulletin of school exchanges. Integrating competence approach is a perspective direction of modern education. 22.* Retrieved from http://visnyk.iatp.org.ua/visnyk/issue_article;22 (In Ukr.).
Пометун О. І. Запровадження компетентнісного підходу – перспективний напрям розвитку сучасної освіти [Електронний ресурс] / О. І. Пометун // Вісник програм шкільних обмінів. – 2004. – № 22. – Режим доступу : http : //visnyk. iatp. org. ua/ visnyk / issue _ article; 22.

Portev, L. & Nikolov, N. (1987). *Methodics of education in mathematics.* Plovdiv, Bulgaria: PU „Paisii Hilendarski”. (In Bul.).
Портев Л. Методика на обучението по математика / Л. Портев, Н. Николов. – Пловдив: ПУ „П. Хилендарски”, 1987.

Portev, L. (2001). Definitness (parametrization) of geometrical figures and applications. In: V. Miloushev (Eds.). *Methods for solving problems* (*from the school course in mathematics*). Part I. (pp. 177-193). Plovdiv, Bulgaria: Makros. (In Bul.).
Портев Л. Определеност (параметризация) на геометричните фигури и приложения / Л. Портев // Методи за решаване на задачи (от училищния курс по математика): Научен ред. В. Милушев. – Ч. І. – Пловдив: Макрос, 2001. – С. 177-193.

Portev, L., Ivanov, I., Milloushev, V., Mavrova, R., Nikolov, Y., Porteva, A., … & Pavlov, H. (2004). *Mathematics – a textbook for the state school-leaving and entrance exams in four parts. Part II „Geometry”.* Plovdiv, Bulgaria: Letera. (In Bul.).
Портев Л. Математика – учебно помагало за държавен зрелостен и кандидатстудентски изпит в четири части. – Част II „Геометрия” / Л. Портев, И. Иванов, В. Милушев, Р. Маврова, Й. Николов, А. Портева, … Х. Павлов. – Пловдив: Летера, 2004.

Portev, L., Mavrova, R., Milloushev, V., Nikolov, N., Kozhuharova, R., Bizova, G. & Makrelov, I. (1996). *Guidance on teaching practicum for the mathematics students of the PU "Paisii Hilendarski".* Plovdiv, Bulgaria: University press „Paisii Hilendarski”. (In Bul.).
Портев Л. Ръководство по учителски практикум за студенти от специалност математика при ПУ "Паисий Хилендарски"/ Л. Портев, Р. Маврова, В. Милушев, Н. Николов, Р. Кожухарова, Г. Бизова, И. Макрело. – Пловдив: ПУИ, 1996.

Poya, D. (1972). *How to solve a problem*. Sofia, Bulgaria: Narodna prosveta. (In Bul.).
Пойа Д. Как да се решава задача / Д. Пойа. – София: Народна просвета, 1972.

Prigodey, N. (1999). *Electrical Engineering Bias and Primary Professional Education in secondary school* (Abstract of the PhD dissertation). Chernihiv state pedagogical university named after Taras Shevchenko, Chernihiv, Ukraine. (In Ukr.).
Пригодій М. А. Профільне та початкове професійне навчання з електротехніки в загальноосвітній школі: автореф. дис. на здобуття наук. ступеня канд. пед. наук: спец. 13.00.02 / Микола Анатолійович Пригодій. – К., 1999. – 20 с.

Pruski, L. & Friedman, J. (2014). An Integrative Approach to Teaching Mathematics, Computer Science, and Physics with Matlab. *Mathematics and Computer Education, 48*(1). Retrieved from https://www.questia.com/library/journal/1P3-3210543981/an-integrative-approach-to-teaching-mathematics-computer

Radko, S. A. (1978). *Developing Cognitive Interest in the Process of Ltarning Activities*. Minsk, Belarus: Nauka i Tekhnika. (In Rus.).
Радько С. А. Формирование интереса к знаниям в процессе учебной деятельности / С. А. Радько. – Минск : Наука и техника, 1978. – 21 с.

Raikov, D. A. (1982). *One-dimensional mathematical analysis.* Moscow, Russia: Vysshaya Shkola. (In Rus.).
Райков Д. А. Одномерный математический анализ : учебное пособие для физ.-мат. фак. пед. ин-тов / Д. А. Райков. – М. : Высшая школа, 1982. – 415 с.

Rakov, S. A. (2005). *Mathematical education: competence approach using ICT. Monograph*. Kharkiv, Ukraine: Fakt. (In Ukr.).
Раков С. А. Математична освіта: компетентнісний підхід з використанням ІКТ : Монографія / С. А. Раков. – Х. : Факт, 2005. – 360 с.

Rihterman, T. D. (1987). *Teaching notions of time to the preschool children.* Moscow, Russia: Prosveshchenie (In Rus.).
Рихтерман Т. Д. Формирование представлений о времени у детей дошкольного возраста / Т. Д. Рихтерман. – М. : Просвещение, 1987. – 47 с.

Riker, P. (2002). *Conflict of interpretations: Essays about hermeneutics.* (I. S. Vdovin, Trans.). Moscow, Russia: «Academia Tsentr» (In Rus.).
Рикёр П. Конфликт интерпретаций: Очерки о герменевтике / П. Рикер / Пер. с фр. И. С. Вдовина. – М. : «Academia-Центр». – 2002. – 415 с.

Roscina, I., Volodko, I. (2014). Using of Computer Algebra Systems in Latvia and Riga Technical University. *Papers of the international scientific conference "Education, science and economics at higher educational establishments and schools. Integration into international educational environment".* Armenia, Cahkadzor, 24-29 March, 2014. Cahkadzor, 511-514. (In Rus.)
Использование систем компьютерной математики в Латвии и в Рижском Техническом университете. *Труды международной научной конференции „Образование, наука и экономика в вузах и школах. Интеграция в международное образовательное пространство"*, Armenia, Цахкадзор, 24-29 March, 2014. Цахкадзор, 511-514.

Rozanova, S. A. (2003). *Mathematical culture of technical university students*. Moscow, Russia: FIZMATLIT. (In Rus.).
Розанова С. А. Математическая культура студентов технических университетов. / С. А. Розанова. – М. : ФИЗМАТЛИТ, 2003. – 176 с.

Rubinstein, S. L. (1973). *Fundamentals of General Psychology*. Moscow, Russia: Pedagogika. (In Rus.).
Рубинштейн С. Л. Проблемы общей психологии / С. Л. Рубинштейн. – М.: Педагогика, 1973. – 424 с.

Rubinstein, S. L. (1999). *Fundamentals of General Psychology.* In 2 parts. (2nd ed.). SPb, Russia: Piter Com. (In Rus.).
Рубинштейн С. Л. Основы общей психологии : в 2 т. / С. Л. Рубинштейн. – 2-е изд. – СПб. : Пиер Ком, 1999. – Т. 1. – 720 с.

Rubtsov, V. V. (1996). Social-psychological conception of the intellectual development of children. A.-N. Perre-Klermon. *Psychological science and education, 2*, 20-26. (In Rus.).
Рубцов В. В. Социально-психологическая концепция интеллектуального развития ребенка А.-Н. Перре-Клермон / В. В. Рубцов // Психологическая наука и образование. – 1996. – № 2. – С. 20–26.

Rukosueva, D. A. (2010). Features of the visualization-aided perception of mathematical knowledge. *Young people and science of the XXI century: Materials of the XI all-Russian (with international participation) theoretical and practical conference of students, postgraduate students, and young scientists*. (Vol. 1). Krasnoyarsk, Russia (In Rus.).
Рукосуева Д. А. Особенности восприятия математических знаний с использованием методов визуализации // Молодежь и наука XXI века: Материалы XI Всерос. науч.-практ. конф. студентов, аспирантов и молодых ученых с международным участием. – Том 1. – Красноярск, 2010. – С. 225–227.

Rules of admission to Cherkasy National University in 2015 (CNU). (2015). Retrieved from http://cdu.edu.ua/info/abiturientam/pryimalna-komisiia/2015-07-07-11-58-5.html. (In Ukr.).
Правила вступу до Черкаського національного університету ім. Б.Хмельницького в 2015 р. : [Електронний ресурс]. Режим доступу: http://cdu.edu.ua/info/abiturientam/pryimalna-komisiia/2015-07-07-11-58-5.html.

Rumelhart, D. E. (1977). *Understanding and summarisation brief stories*. *Basic processes in reading: perceptions and comprehension.* New Jersey.

Russell, B. (1948). *Human Knowledge, Its Scope and Limits*. London: George Allen & Unvin.

Rybin, V. A. (2012). *University of the XXIst century: Anthropological prospects of education and culture*. Moscow, Russia: Librocom. (In Rus.).
Рыбин В. А. Университет XXI века: Антропологические перспективы образования и культуры. / В. А. Рыбин. – М. : Книжный дом «Либроком», 2012. – 176 с.

Safonova, I. Y. (2014). Mathematical competence as a factor in preparing the high school students for life. *The current problems of public administration, education and psychology*, *2*, 65-68. (In Ukr.).
Сафонова І. Я. Математична компетентність як фактор підготовки старшокласників до життя / І. Я. Сафонова // Актуальні проблеми державного управління, педагогіки та психології. – 2014. – Вип. 2. – С. 65-68.

Savchenko, O. Ya. (2012). Learning environment as a factor of stimulating the research activity of primary school children. *Scientific notes Minor Academy of Sciences of Ukraine, 1*, 41-49. (In Ukr.).
Савченко О. Я. Навчальне середовище як чинник стимулювання дослідницької діяльності молодших школярів / О. Я. Савченко // Наукові записки Малої академії наук України. – 2012. – №. 1. – С. 41-49.

Savenkov, A. I. (2004). Research training and engineering in modern education. *Research work of students, 1*, 22-32. (In Rus.).
Савенков А. И. Исследовательское обучение и проектирование в современном образовании / А. И. Савенков // Исследовательская работа школьников. – 2004. – № 1. – С.22-32.

Sazonova, A. V. (2010). *General theoretic bases of natural and mathematical educations of the preschool children.* Kiev, Ukraine: Vidavnichiy dim «Slovo» (In Ukr.).
Сазонова А. В. Загальнотеоретичні основи природничо-математичної освіти дітей дошкільного віку / А. В.Сазонова. – К. : Видавничий дім «Слово», 2010. – 248 с.

Scafa, E., Vlasenko, E., & Goncharova, I. (2003). *An integrated approach to the development of a creative personality through a system of heuristic mathematics tasks*. Donetsk, Ukraine: Company TEAH. (In Ukr.).
Скафа Е. Комплексный подход к развитию творческой личности через систему эвристических заданий по математике : книга для учителя: на материале 7 класса / Е. Скафа, Е. Власенко, И. Гончарова. – Донецк : Фирма ТЕАН, 2003. – 204 с.

Schuepbach, M. (2015). Effects of Extracurricular Activities and Their Quality on Primary School-Age Students' Achievement in Mathematics in Switzerland. *School Effectiveness and School Improvement: An International Journal of Research, Policy and Practice, 26* (2), 279-295. doi: 10.1080/09243453.2014.929153

Seemann, E. (2015). Unit testing Maths automated assessment of mathematic exercises. *Lecture Notes in Computer Science*, *9307*, 530-534.

Semenovich, A.F. (1961). *Geometry. Trial textbook for the sixth grade*. Ulyanovsk, Russia. (In Rus.).
Семенович А. Ф. Геометрия. Пробный учебник для шестого класса / А. Ф. Семенович. – Ульяновск, 1961. – 164 с.

Semenovich, A.F. (1962). *Geometry. Trial textbook for the seventh grade*. Ulyanovsk, Russia. (In Rus.).
Семенович А. Ф. Геометрия. Пробный учебник для седьмого класса / А. Ф. Семенович. – Ульяновское книжное издательство, 1962. – 94 с.

Semenovich, A.F. (1963). *Geometry. Trial textbook for the eighth grade*. Ulyanovsk, Russia. (In Rus.).
Семенович А. Ф. Геометрия. Пробный учебник для восьмого класса : пособие для учителей / А. Ф. Семенович. – Ульяновское книжное издательство, 1963. – 90 с.

Serova, S. O., & Fomina, N. V. (2006). Way into the world of scientific technology. *School management, 3*, 27-29. (In Ukr.).
Сєрова С. О., Фоміна Н. В. Шлях у світ наукових технологій / С. О. Сєрова, Н. В. Фоміна // Управління школою – 2006. – №3. – С. 27-29.

Shamova, T. I. (1982). *Enhancing school teaching*. Moscow, Russia: Pedagogika. (In Rus.).
Шамова Т. И. Активизация учения школьников / Т. И. Шамова. – М.: Педагогика, 1982. – 208 с.

Sharigin, I. F. (1989). We are learning to solve geometry problems. *Mathematics in school, 2*, 87-101. (In Rus.).
Шарыгин И. Ф. Учимся решать задачи по геометрии / И. Ф. Шарыгин // Математика в школе. – 1989. – № 2. – С. 87-101.

Shcherbakova, K. Y. (2011). *Methods of teaching elements of mathematics to the children of the preschool age.* Kyiv, Ukraine. (In Ukr.).
Щербакова К. Й. Методика формування елементів математики у дітей дошкільного віку / К. Й. Щербакова. – Київ, 2011. – 362 с.

Shchukina, G. I. (1988). *Pedagogical Problems of Developing Students' Cognitive Interests*: *Scientific Edition*. Moscow, Russia: Pedagogika. (In Rus.).
Щукина Г. И. Педагогические проблемы формирования познавательных интересов учащихся : научное издание / Г. И. Щукина ; АПН СССР. – М. : Педагогика, 1988. – 208 с.

Shchukina, G. I. (1986). *Role activities in the learning process.* Moscow, Russia: Prosveschenie. (In Rus.).
Щукина Г И. Роль деятельности в учебном процессе : [кн. для учителя] / Г. И. Щукина. – М.: Просвещение, 1986. – 144 с.

Shelestova, L. (Ed.). (2003). *How to help your child become a creative person.* Kyiv, Ukraine: Shkil'nyy svit. (In Ukr.).
Як допомогти дитині стати творчою особистістю/ Упорядн. Л. Шелестова. – К.: Шкільний світ, 2003. – 112с.

Sheptulin, A. P. (1967). *The system of dialectic categories*. Moscow, Russia: Nauka. (In Rus.).
Шептулин А. П. Система категорий диалектики / А. П. Шептулин. – М.: Наука, 1967. – 376 с.

Shkil, M. I., Slyepkan, Z. I., & Dubynchuk, O. S. (2006). *Algebra and Introduction into Analysis. Grade 11: Tutorial*. Kiev, Ukraine: Zodiak – ECO. (In Ukr.).
Шкіль М. І. Алгебра і початки аналізу. 11 клас : підручник / Шкіль М. І., Слєпкань З. І., Дубинчук О. С. – К. : Зодіак–Еко, 2006. – 384 с.

Shtern, V. (2009). Psychology of early childhood and up to six years of age. In E. Minkov (Ed.), *History of foreign psychology from the end of XIX till begining of XX centuries: Reading-book* (p.p. 80-87). Moscow, Russia: Flynta: MPSK (In Rus.).
Штерн В. Психология раннего детства до шестилетнего возраста / В. Штерн. – М., 2009 // История зарубежной психологии конца XIX-начала XX в.: хрестоматія / Под ред. Е. С. Минькова. – М. : Флинта : МПСК, 2009. 136 с. – С. 80–87.

Shtoff, V. A. (1963). *The role of models in cognition*. Leningrad, Russia: Izdatelstvo Leningradskogo universiteta. (In Rus.).
Штофф В. А. Роль модели в познании / В. А. Штофф. – Л.: Изд-во Ленингр. ун-та, 1963. – 128 с.

Shumakova, N. B. (1986). Research activity in the form of questions in different age periods. *Voprosy psihologii, 1*, 27-32. (In Rus.).
Шумакова Н. Б. Исследовательская активность в форме вопросов в разные возрастные периоды / Н. Б. Шумакова // Вопросы психологии. – 1986. – №1.

Shumyhai, S. M. (2013). *Developing Cognitive Interest of High School Students in Teaching Mathematics through the History of Science.* (Unpublished PHD dissertation) M. P. Dragomanov National Pedagogical University, Kyiv, Ukraine. (In Ukr.).
Шумигай С. М. Розвиток пізнавального інтересу учнів основної школи до вивчення математики засобами історії науки : дис. … канд. пед. наук : спец. 13.00.02 «Теорія та методика навчання (математика)» / Світлана Миколаївна Шумигай; Нац. пед. ун-т імені М. П. Драгоманова. – К., 2013. – 274 с.

Shyian, N. (2005). *Didactic Bases of Profile Education in Rural Schools* (Abstract of the Doctoral dissertation). Poltava State Pedagogical University named after V. Korolenko, Poltava, Ukraine. (In Ukr.).
Шиян Н. І. Дидактичні засади профільного навчання у загальноосвітній школі сільської місцевості: автореф. дис. … д-ра пед. наук: спец. 13.00.09 / Надія Іванівна Шиян. – Харків, 2005. – 40 с.

Simakov, M. Yu. (2008). *Mathematics and the world*. Moscow, Russia: Samoobrazovanie. (In Rus.).
Симаков М. Ю. Математика и мир. / М. Ю. Симаков. – М. : Самообразование, 2008. – 96 с.

Skafa, E. & Milloushev, V. (2009). *Constructing educational-cognitive heuristic activity in solving mathematical problems.* Plovdiv, Bulgaria: University press „Paisii Hilendarski”. (In Bul.).
Скафа Е. Конструиране на учебно-познавателна евристична дейност по решаване на математически задачи / Е. Скафа, В. Милушев. – Пловдив: УИ „Паисий Хилендарски”, 2009.

Skatkin, M. N. (1986). *Methodology and methods of educational research*. Moscow, Russia: Pedagogika. (In Rus.).
Скаткин М. Н. Методология и методика педагогических исследований / М. Н. Скаткин. – М.: Педагогика, 1986. – 152 с.

Skatkin, M. N. (Ed.). (1982). *Didactics of High School: Some problems of modern didactics* (2th ed.). Moscow, Russia: Prosveshchenie. (In Rus.).
Дидактика средней школы: Некоторые проблемы современной дидактики / Под ред. М. Н. Скаткина. – 2-е изд., перераб. и доп. – М. : Просвещение, 1982.– 319 с.

Skvortsova, S. O. (2006). *Methodical system of teaching primary school pupils how to solve story problems. Monograph*. Odesa, Ukraine: Astroprynt. (In Ukr.).
Скворцова С. О. Методична система навчання розв’язування сюжетних задач учнів початкових класів: Монографія. – Одеса: Астропринт, 2006. – 696 с.

Slavov, K. (1969). *Basic methods for solving problems in algebra*. Sofia, Bulgaria: Narodna prosveta. (In Bul.)
Славов К. Основни методи за решаване на задачи по алгебра / К. Славов. – София: Народна просвета, 1969.

Sliepkan, Z. I. (2000). *Methods of teaching mathematics*. Kyiv, Ukraine: Zodiak-ECO. (In Ukr.).
Слєпкань З. І. Методика навчання математики : підруч. для студ. мат. спеціальностей пед. навч. закл. / З. І. Слєпкань. – К. : Зодіак-ЕКО, 2000. – 512 с.

Sliepkan, Z. I. (2004). *Psycho-pedagogical and methodological foundations of developing teaching mathematics*. Ternopil, Ukraine: Pidruchnyky i posibnyky. (In Ukr.).
Слєпкань З. І. Психолого-педагогічні та методичні основи розвивального навчання математики / З. І. Слєпкань. – Тернопіль: Підручники і посібники, 2004. – 240с.

Smolentseva, A. A. (1993). *Story-based didactic games in mathematics.* Moscow, Russia: Prosveshchenie (In Rus.).
Смоленцева А. А. Сюжетно-дидактическая игра с математическим содержанием / А. А. Смоленцева. – М. : Просвещение, 1993. – 97 с.

Starchenko, V. A. (2009). *Developing logical-mathematical competence in older preschoolers*. Kiev, Ukraine: Svitych. (In Ukr.).
Старченко В. А. Формування логіко-математичної компетентності у старших дошкільників: [навч. – мет. пос. до базової програми розв. д. «Я у Світі».] – К. : Світич, 2009. – 80 с.

State national program "Education. Ukraine XXI century". (1994). Kyiv, Ukraine: Raduha. (In Ukr.).
Державна національна програма «Освіта. Україна XXI століття». – К.: Райдуга, 1994. – 61 с.

State target social program of upgrading school natural and mathematical education for the period till 2015. (2011). Retrieved from http://dniokh.gov.ua/wp-content/uploads/2014/12/Prirodnicho-matematichna.pdf (In Ukr.).
Державна цільова соціальна програма підвищення якості шкільної природничо-математичної освіти на період до 2015 року [Електронний ресурс]. – 2011. – Режим доступу: http://dniokh.gov.ua/wp-content/uploads/2014/12/ /Prirodnicho-matematichna.pdf

Stepanova, T. M. (2006). *Individualization and differentiation of teaching mathematics to children of the senior preschool age.* Kiev, Ukraine: Vidavnichiy dim «Slovo» (In Ukr.).
Степанова Т. М. Індивідуалізація і диференціація навчання математики дітей старшого дошкільного віку : [монографія] / Т. М. Степанова. – Київ : Видавничий дім «Слово», 2006. – 208 с.

Stillman, G. A., Kaiser, G., Blum, W. & Brown, J. P. (Eds.). (2013). *Teaching Mathematical Modelling: Connecting to Research and Practice*. Netherlands: Springer.

Stolyar, A. (1986). *Psychology of Mathematics: [lectures]*. Minsk, Belarus: Vysheyshaya shkola (In Rus.).
Столяр А. А. Педагогика математики: [курс лекций] / А. А. Столяр. – [изд. 3-е перераб. и доп.]. – Минск: Вышейшая школа, 1986. – 414 с.

Stolyar, A. A. (Ed.). (1988). *Development of the elementary mathematical notions of preschoolers.* Moscow, Russia: Prosveshchenie (In Rus.).
Формирование элементарных математических представлений у дошкольников / Под ред. А. А. Столяра. – М. : Просвещение, 1988. – 303 с.

Sushchenko, T. I. (1970). *Fostering Cognitive Interests of Adolescents in Extracurricular Activity.* Kyiv, Ukraine: Radianska Shkola. (In Ukr.).
Сущенко Т. І. Виховання пізнавальних інтересів у підлітків в позашкільній роботі / Т. І. Сущенко. – К. : Рад. школа, 1970. – 96 с.

Talyzina, N. F. (1983). *Developing cognitive activity of students*. Moscow, Russia: Znanie. (In Rus.)
Талызина Н. Ф. Формирование познавательной деятельности учащихся / Н. Ф. Талызина. –Москва: Знание, 1983.

Talyzina, N. (1998). *Pedagogical psychology*. Moscow, Russia: Prosvieshcheniie. (In Rus.).
Талызина Н. Ф. Педагогическая психология : учеб. пособ. [для студ. сред. пед. учеб. завед.] / Нина Федоровна Талызина. – М. : Изд. центр «Академия», 1998. – 288 с.

Talyzina, N. F. (2007). The essence of the activity approach in psychology. *Methodology and History of Psychology, 4*, 157-162.
Талызина Н. Ф. Сущность деятельностного подхода в психологии / Н. Ф. Талызина // Методология и история психологии. – 2007. – № 4. – С. 157-162.

Tarasenkova, N. A. (1991). *Activation of cognitive activity of students in a lecture and practical mathematics learning in school* (Unpublished Doctoral dissertation). Kiev state pedagogical university, Kiev, Ukraine. (In Rus.).
Тарасенкова Н. А. Активизация познавательной деятельности учащихся в условиях лекционно-практической системы обучения математике в школе: дис. ... канд. пед. н. : 13.00.02 / Тарасенкова Нина Анатольевна. – Киев, 1991. – 211 с.

Tarasenkova, N. A. (2002). *Using sign and symbolic means in teaching mathematics. Monograph.* Cherkasy, Ukraine: Vidlunnya-Plyus. (In Ukr.).
Тарасенкова Н. А. Використання знаково-символічних засобів у навчанні математики / Н. А. Тарасенкова. – Черкаси : Відлуння-Плюс, 2002. – 400 с.

Tarasenkova, N. A. (2003). *The theoretic-methodical principles of using the sign and symbolic means in teaching mathematics to the basic school students* (Unpublished Doctoral dissertation), Bohdan Khmelnytsky National University at Cherkasy, Cherkasy, Ukraine. (In Ukr.).
Тарасенкова Н. А. Теоретико-методичні основи використання знаково-символьних засобів у навчанні математики учнів основної школи : дис. ... д-ра пед. н. : 13.00.02 / Тарасенкова Ніна Анатоліївна. – Черкаси, 2003. – 630 с.

Tarasenkova, N. A. & Kirman, V. K. (2008). The content and structure of mathematical competence of the secondary school students. *Mathematics in school, 6*, 3-9. (In Ukr.).
Тарасенкова Н. А. Зміст і структура математичної компетентності учнів загальноосвітніх навчальних закладів / Н. А. Тарасенкова, В. К. Кірман // Математика в школі. – 2008. – № 6. – С. 3–9.

Tarasenkova, N. A. (2013). Specifications of coding geometric concepts. *Science and Education a New Dimension: Pedagogy and Psychology*, *5*, 7-11. (In Rus.)
Тарасенкова Н. А. Особенности кодирования геометрических понятий / Н. А. Тарасенкова // Science and Education a New Dimension: Pedagogy and Psychology. – 2013. – №5. – С. 7-11.

Tarasenkova, N. (2013). The quality of mathematical education in the context of Semiotics. *American Journal of Educational Research, 1*(11), 464-471. doi: 10.12691/education-1-11-2.

Tarasenkova, N. A, Bogatyreva, I. M., Bochko, O. P., Kolomiets, O. M., & Serdiuk, Z. O. (2013). Conceptual principles of development of text-books on mathematics for 5–6 classes. *Science and education a new dimension*, *2* (Marhc), 34-38. Retrieved from http://seanewdim.com/published-issues.html (In Ukr.).

Tarasenkova, N. A, Bogatyreva, I. M., Bochko, O. P., Kolomiets, O. M., & Serdiuk, Z. O. (2013). *Mathematics*: *textbook for the 5th form for the secondary schools*. Kyiv, Ukraine: Publishing House "Osvita". (In Ukr.).
Тарасенкова Н. А. Математика : [підруч. для 5 кл. загальноосв. навч. закл.] / Н. А. Тарасенкова, І. М. Богатирьова, О. П. Бочко, О. М. Коломієць, З. О. Сердюк. – К. : Видавничий дім "Освіта", 2013. – 352 с.

Tarasenkova, N. A., Serdyuk, Z. O. (2013). The basis of comparative pedagogy in mathematical education of different countries investigation. *Didactics of mathematics: problems and research: international collection of scientific works, 40*, 55–59 (In Ukr.).
Тарасенкова Н. А. Основи порівняльної педагогіки у дослідженні шкільної математичної освіти різних країн / Н. А. Тарасенкова, З. О. Сердюк // Дидактика математики: проблеми і дослідження: міжнародний збірник наукових робіт. – Донецьк: Вид-во ДонНУ, 2013. – Вип. 40. – С. 55–59.

Tarasenkova, N. (2014). Peculiar Features of Verbal Formulations in School Mathematics. *Global Journal of Human-Social science : G : Linguistics & Education*, *14*(3), 61-67. Retrieved from http://globaljournals.org/journals/human-social-science/g-linguistics-education

Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M., & Serdiuk, Z. O. (2014). *Mathematics*: *textbook for the 6th form for the secondary schools.* Kyiv, Ukraine: Publishing House "Osvita". (In Ukr.).
Тарасенкова Н. А. Математика : [підруч. для 6 кл. загальноосв. навч. закл.] / Н. А. Тарасенкова, І.М. Богатирьова, О.М. Коломієць, З.О. Сердюк. – К. : ВД "Освіта", 2014. – 304 с.

Tarasenkova, N. (2015). Non-verbal Shells of the Instructional Mathematical Content. *American Journal of Educational Research*, *3*(12B), 1-5. doi: 10.12691/education-3-12B-1.

Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M., & Serdiuk, Z. O. (2015). *Algebra*: *textbook for the 7th form for the secondary schools*, Kyiv, Ukraine: Publishing House "Osvita". (In Ukr.).
Тарасенкова Н. А. Алгебра : [підруч. для 7 кл. загальноосв. навч. закл.] / Н. А. Тарасенкова, І. М. Богатирьова, О. М. Коломієць, З. О. Сердюк. – К. : Видавничий дім "Освіта", 2015. – 304 с.

Tarasenkova, N. A., Bogatyreva, I. M, Kolomiets, O. M., Serdiuk, Z. O. (2015). *Test subject competences. Mathematics, 5th grade of the secondary school. Collection of tasks for assessment of pupil achievements* [Teach. method. guidances.]. Kyiv, Ukraine: Orion. (In Ukr.).
Тарасенкова Н. А. Перевірка предметних компетентностей. Математика, 5 кл. Збірник завдань для оцінювання навчальних досягнень учнів: [навч.-метод. посіб.] / Н. А. Тарасенкова, І. М. Богатирьова, О. М. Коломієць, З. О. Сердюк; за ред. Н. А. Тарасенкової. – К.: Оріон, 2015. – 48 с.

Tarasenkova, N. A., Bogatyreva, I. M, Kolomiets, O. M., Serdiuk, Z. O. (2015). *Test subject competences. Mathematics, 6th grade of the secondary school. Collection of tasks for assessment of pupil achievements* [Teach. method. guidances.]. Kyiv, Ukraine: Orion. (In Ukr.).
Тарасенкова Н. А. Перевірка предметних компетентностей. Математика, 6 кл. Збірник завдань для оцінювання навчальних досягнень учнів: [навч.-метод. посіб.] / Н. А. Тарасенкова, І. М. Богатирьова, О. М. Коломієць, З. О. Сердюк; за ред. Н. А. Тарасенкової. – К.: Оріон, 2015. – 40 с.

Tarasenkova, N. A., Burda, M. I., Bogatyreva, I. M, Kolomiets, O. M., Serdiuk, Z. O. (2015). *Test subject competences. Geometry, 7th grade of the secondary school. Collection of tasks for assessment of pupil achievements* [Teach. method. guidances]. Kyiv, Ukraine: Orion. (In Ukr.).
Тарасенкова Н. А. Перевірка предметних компетентностей. Геометрія, 7 кл. Збірник завдань для оцінювання навчальних досягнень учнів: [навч.-метод. посіб.] / Н. А. Тарасенкова, М. І. Бурда, І. М. Богатирьова, О. М. Коломієць, З. О. Сердюк; за ред. Н. А. Тарасенкової. – К.: Оріон, 2015. – 24 с.

Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M. & Serdyik, Z. O. (2015). The structure and content of teaching kits in algebra for the 7th Form. *Science and education a new dimension III (26),* 64, 12-18. (In Ukr.).
Тарасенкова Н. А. Засоби перевірки математичної компетентності в основній школі / Н. А. Тарасенкова, І. М. Богатирьова, О. М. Коломієць, З. О. Сердюк // Science and education a new dimension / Chief Honorary Editor: N. Tarasenkova. – III (26), Issue: 64. – Budapest: SCASPEE, 2015. – P. 12−18.

Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M. & Serdyik, Z. O. In N. A. Tarasenkova (Ed.). (2015). *Verification of subject competencies. Mathematics, Grade 5. Collection of tasks to the evaluation of educational achievements of students*. Kyiv, Ukraine: Orion. (In Ukr.).
Тарасенкова Н. А. Перевірка предметних компетентностей. Математика, 5 кл. Збірник завдань для оцінювання навчальних досягнень учнів: [навч.-метод. посіб.] / Н. А. Тарасенкова, І. М. Богатирьова, О. М. Коломієць, З. О. Сердюк; за ред. Н. А. Тарасенкової. – К.: Оріон, 2015. – 48 с.

Tarasenkova, N. A., Bogatyreva, I. M., Kolomiets, O. M. & Serdyik, Z. O. In N. A. Tarasenkova (Ed.). (2015). *Verification of subject competencies. Mathematics, Grade 6. Collection of tasks to the evaluation of educational achievements of students*. Kyiv, Ukraine: Orion. (In Ukr.).
Тарасенкова Н. А. Перевірка предметних компетентностей. Математика, 6 кл. Збірник завдань для оцінювання навчальних досягнень учнів: [навч.-метод. посіб.] / Н. А. Тарасенкова, І. М. Богатирьова, О. М. Коломієць, З. О. Сердюк; за ред. Н. А. Тарасенкової. – К.: Оріон, 2015. – 40 с.

Tarasenkova, N. A., Burda, M. I., Bogatyreva, I. M., Kolomiets, O. M. & Serdyik, Z. O. In N. A. Tarasenkova (Ed.). (2015). *Verification of subject competencies. Geometry, Grade 7. Collection of tasks to the evaluation of educational achievements of students*. Kyiv, Ukraine: Orion. (In Ukr.).
Тарасенкова Н. А. Перевірка предметних компетентностей. Геометрія, 7 кл. Збірник завдань для оцінювання навчальних досягнень учнів: [навч.-метод. посіб.] / Н. А. Тарасенкова, М. І. Бурда, І. М. Богатирьова, О. М. Коломієць, З. О. Сердюк; за ред. Н. А. Тарасенкової. – К.: Оріон, 2015. – 24 с.

Tarasenkova, N. A., Globin, O. I., Bogatyreva, I. M., Kolomiets, O. M. & Serdyik, Z. O. In N. A. Tarasenkova (Ed.). (2015). *Verification of subject competencies. Algebra, Grade 7. Collection of tasks to the evaluation of educational achievements of students*. Kyiv, Ukraine: Orion. (In Ukr.).
Тарасенкова Н. А. Перевірка предметних компетентностей. Алгебра, 7 кл. Збірник завдань для оцінювання навчальних досягнень учнів: [навч.-метод. посіб.] / Н. А. Тарасенкова, О. І. Глобін, І. М. Богатирьова, О. М. Коломієць, З. О. Сердюк; за ред. Н. А. Тарасенкової. – К.: Оріон, 2015. – 32 с.

Tarasenkova, N. A., Bogatyreva, I. M, Kolomiets, O. M., Serdiuk, Z. O. (2016). *Algebra. Textbook for the 8th grade of the secondary school.* Kyiv, Ukraine: Orion. (In Ukr.).
Тарасенкова Н. А. Алгебра : [підруч. для 8 кл. загальноосв. навч. закл.] / Н.А. Тарасенкова, І.М. Богатирьова, О.М. Коломієць, З.О. Сердюк. – К. : УОВЦ "Оріон", 2016. – 336 с.

Taruntaeva, T. V. (1980). *The development of elementary mathematical concepts in preschoolers.* Moscow, Russia: Prosveshchenie (In Rus.).
Тарунтаева Т. В. Развитие элементарных математических представлений у дошкольников / Т. В.Тарунтаева. – М. : Просвещение, 1980. – 62 с.

Tatarinova, S. O. (2008). *Developing logical-mathematical concepts in senior preschoolers in cognitive activity.* (Unpublished Doctoral dissetation). South Ukrainian National Pedagogical University named after K. D. Ushynsky, Odessa, Ukraine. (In Ukr.).
Татаринова С. О. Формування логіко-математичних понять у старших дошкільників у процесі пізнавальної діяльності : дис. … канд. пед. наук: спец. 13.00.08 / С. О.Татаринова. – Одеса, Південноукраїнський державний педагогічний університет імені К. Д. Ушинського, 2008. – 205 с.

Testov, V. A. (2011). Selection of content of teaching mathematics. *Mathematical methods and models: theory, applications and role in education: collection of scientific works*, 265-273 (In Rus.).
Тестов В. А. Отбор содержания обучения математике / В. А. Тестов // Математические методы и модели: теория, приложения и роль в образовании : сборник научных трудов. – Ульяновск : УЛГТУ, 2011. – С. 265-273.

Testov, V. A. (2015). The main aspects of the implementation of the concept of development of mathematics education in the Russian Federation. *Innovative projects and programs in education, 1*. Retrieved from http://cyberleninka.ru/article/n/osnovnye-aspekty-realizatsii-kontseptsii-razvitiya-matematicheskogo-obrazovaniya-v-rossiyskoy-federatsii (In Rus.).
Тестов В. А. Основные аспекты реализации концепции развития математического образования в Российской Федерации / В. А. Тестов // Инновационные проекты и программы в образовании. – Выпуск № 1. – 2015 : [Электронный ресурс]. Режим доступа: http://cyberleninka.ru/article/n/osnovnye-aspekty-realizatsii-kontseptsii-razvitiya-matematicheskogo-obrazovaniya-v-rossiyskoy-federatsii

The Law of Ukraine "On General Secondary Education". Retrieved from http://zakon2.rada.gov.ua/laws/show/651-14. (In Ukr.).
Закон України «Про загальну середню освіту» [Електронний ресурс]. – Режим доступу : http://zakon2.rada.gov.ua/laws/show/651-14.

The Ukrainian Standardized External Testing Initiative (USETI) Final Report. April 2007 – December 2009. Retrieved from http://pdf.usaid.gov/pdf_docs/Pdacq648.pdf.

Tiheeva, E. & Morozova, M. (1927). *Counting in the life of little children.* Moscow – Leningrad, Russia: Gosizdat (In Rus.).
Тихеева Е. Счет в жизни маленьких детей / Е. Тихеева, М. Морозова. – Москва – Ленинград : Госиздат, 1927. – 67 с.

Tonov, I. (2005). Educating mathematics teachers through solving problems. *Mathematics and informatics*, *3*, 5-10. (In Bul.)
Тонов И. Обучението на учителите по математика чрез решаване на задачи / И. Тонов // Математика и информатика. – 2005. – №3. – С. 5-10.

Ukrainian Centre for Educational Quality Assessment (UCEQA). (2015). Report on external independent testing of learning outcomes for Senior Secondary School graduates who wish to enter higher education in Ukraine in 2015. Retrieved from http://testportal.gov.ua/reports.
УЦОЯО. (2015). Звіт про прове-дення зовнішнього незалежного оцінювання результатів навчання, здобутих на основі повної загальної середньої освіти, для осіб, які виявили бажання вступа-ти до вищих навчальних закладів України в 2015 році : [Електронний ресурс]. Режим доступу: http://testportal.gov.ua/reports.

Ukrainian Centre for Educational Quality Assessment (USEQUA). (2015). Syllabus of External Independent Assessment in Mathematics. Retrieved from http://testportal.gov.ua/prepare_math/
Український центр оцінювання якості освіти (УЦОЯО). (2015). Програма ЗНО з математики : [Електронний ресурс]. Режим доступу: http://testportal.gov.ua/ /prepare_math/

Usova, A. P. (1981). *Teaching in kindergarten.* Moscow, Russia: Prosveshchenie (In Rus.).
Усова А. П. Обучение в детском саду / А. П. Усова. – М. : Просвещение, 1981. – 176 с.

Vasilev, V. (2006). *Reflection in knowledge, self-knowledge and practice*. Plovdiv, Bulgaria: Makros. (In Bul.)
Василев В. Рефлексията в познанието, самопознанието и практиката / В. Василев. – Пловдив: Макрос, 2006.

Vasilev, V., Dimova, Y. & Kolarova-Kancheva, T. (2005). *Reflection and learning. Part 1: Reflection – theory and practice*. Plovdiv, Bulgaria: Makros. (In Bul.)
Василев В. Рефлексия и обучение. Част 1: Рефлексията – теория и практика / В. Василев, Й. Димова, Т. Коларова-Кънчева. – Пловдив: Макрос, 2005.

Vasilevskii, A. B. (1988). *Training in solving problems in mathematics*. Minsk, Belarus: Visheishaya shkola.
Василевский А. Б. Обучение решению задач по математике / А. Б. Василевс-кий. – Минск: Вышейшая школа, 1988.

Vasylenko, I. O. (2015). *Development of Basic School Students' Cognitive Interest in Terms of Extracurricular Activities in Mathematics.* (Unpublished doctoraldissertation). Bohdan Khmelnytsky National University of Cherkasy, Cherkasy, Ukraine. (In Ukr.).
Василенко І. О. Формування пізнавального інтересу учнів основної школи у позаурочній роботі з математики: дис. ... канд. пед. наук : спец. 13.00.02 «Теорія і методика навчання (математика)» / Ірина Олександрівна Василенко ; Черкас. нац. ун-т імені Богдана Хмельницького. – Черкаси, 2015. – 288 с.

Verbytsky, A. A. (1991). *Active learning in High School: context approach: educational tutorial*. Moscow, Russian: Vysshaya Shkola. (In Rus.).
Вербицкий А. А. Активное обучение в высшей школе: контекстный подход : метод. пособие / А. А. Вербицкий. – М. : Высшая школа, 1991. – 207 с.

Verkhovna Rada of Ukraine. (1999). About General Secondary Education (Law of Ukraine). Retrieved from zakon.rada.gov.ua/laws/show/651-14 (In Ukr.).
Закон України «Про загальну середню освіту» [Електронний ресурс] // Відомості Верховної Ради України. – 1999. – №28. – С. 230. – Режим доступу: zakon.rada.gov.ua/laws/show/651-14

Vigotskii, L. S. (1956). *Selected psychological studies.* Moskva, Russia.
Выготский Л. С. Избранные психологические исследования / Л. С. Выготский. – Москва, 1956.

Volkova, N. P. (2003). *Pedagogy: A Handbook for students. Higher teach. bookmark.* Kyiv, Ukraine: Publishing Center "Academy" (In Ukr.).
Волкова Н. П. Педагогіка : посібник для студ. вищих навч. закл. / Н. П. Вол-кова. – К. : Видавничий центр "Академія", 2003. – 576 с. - (Альма-матер).

Volodko, I., Dzenite, I. (2009). The Use of the Package "Mathematica" in Teaching of Mathematics in Riga Technical Univ Roscina, I., Volodko, I. (2014)ersity. *Proceedings of the 2009 International Conference on Engineering and Mathematics*, Spain, Bilbao, 17-19 June, 2009. Bilbao, Spain: Javier Bilbao Landatxe, 75-79.

Volodko, I., Roscina, I. (2012). The Use of ORTUS Portal for Teaching Mathematics in Riga Technical University. *Some topical problems of the modern mathematics and mathematical education: scientific conference materials "Gercenovskis readings – 2012",* Russia, SanktPeterburg, 16-21 April, 2012. SanktPeterburg: *the Library of the Academy of Science*, 168-173. (In Rus.).
Использование портала ORTUS для обучения математике в Рижском Техническом университете. *Некоторые актуальные проблемы современной математики и математического образования: материалы научной конференции "Герценовские чтения - 2012"*, Russia, Санкт Петербург, 16-21 April, 2012. Санкт Петербург: Библиотека Академии наук, 168-173.

Volodko, I., Roscina, I. (2014). Using the Package MATHEMATICA for Teaching Mathematics in Riga Technical University. *Papers of the international scientific conference "Education, science and economics at higher educational establishments and schools. Integration into international educational environment", A*rmenia, Cahkadzor, 24-29 March, 2014. Cahkadzor, 482-485. (In Rus.).
Использование пакета программ Mathematica9 при обучении математике в Рижском Техническом университете. *Труды международной научной конференции „Образование, наука и экономика в вузах и школах. Интеграция в международное образовательное пространство"*, Armenia, Цахкадзор, 24-29 March, 2014. Цахкадзор, 482-485.

Volodko, I., Cerņajeva, S. (2016). Assessment of Students' Knowledge by Means of Tests. *Rural Environment. Education. Personality. (REEP): Proceedings of the 9th International Scientific Conference*, Latvia, Jelgava, 13-14 May, 2016. Jelgava: Latvia University of Agriculture, 328-333.

Volovyk (Bochko), O. (2010). *Methodological principles of organization and effectuation of practicum on solving mathematical problems in classical universities*. (Unpublishing Doctoral dissertation). Bohdan Khmelnytsky National University of Cherkasy, Cherkasy, Ukraine. (In Ukr.).
Воловик (Бочко) О. П. Методичні засади організації та проведення практикуму з розв'язування математичних задач у класичних університетах: дис. … канд. пед. наук. 13.00.02 – теорія та методика навчання (математика) / Оксана Петрівна Воловик (Бочко). – Черкаси: Черкаський національний університет імені Богдана Хмельницького, 2010. – 222 с.

Volyanskaya, O. (1999). *Algebra and elementary analysis in the professional-technical school under the educational standard implementation* (Abstract of the PhD dissertation). M. P. Dragomanov National pedagogical University, Kyiv, Ukraine. (In Ukr.).
Волянська О. Є. Вивчення алгебри і початків аналізу в професійно-технічних училищах в умовах впровадження освітнього стандарту: автореф. дис... канд. пед. наук: 13.00.02 / О. Є. Волянська – К. : Нац. пед. ун-т ім. М.П.Драгоманова., 1999. – 18 с.

Voronina, L. A. (2011). *Mathematical education in the period of preschool childhood: planning methodology.* (Abstract of the Doctoral dissertation). Urals State Pedagogical University, Yekaterinburg, Russia (In Rus.).
Воронина Л. А. Математическое образование в период дошкольного детства: методология проектирования : автореф. дис.… д-ра пед. наук: спец. 13.00.02 / Л. А. Воронина – Екатеринбург, Уральский государственный педагогический университет, 2011. – 47 с.

Ward-Penny, R. (2011). *Cross-Curricular Teaching and Learning in the Secondary School Mathematics* (1st ed.). London: Routledge.

Willcox, K., Bounova, G. (2004). *Mathematics in engineering: Identifying, enhancing and linking the implicit mathematics curriculum.* Paper presented at 2004 Annual Conference of American Society for Engineering Education. Annual Conference&Exposition, Salt Lake Sity, USA.

Yakymanska, I. S. (2000). *Technology of Personality-Oriented Teaching in Modern School.* Moscow, Russia: Sentyabr'. (In Rus.).
Якиманская И. С. Технология личностно-ориентированного обучения в современной школе / И. С. Якиманская. – М. : Сентябрь, 2000. – 176 с.

Yesayan, R. S. (2005). *Developing Adolescents' Interest in Self-Education in Extracurricular Activity.* (Unpublishing Doctoral dissertation). Institute of Preschool Education, Moscow, Russia. (In Rus.).
Есаян Р. С. Формирование у школьников-подростков интереса к самообразованию во внеурочной работе : дисс. ... канд. пед. наук : спец. 13.00.01 «Общая педагогика, история педагогики и образования» / Рудольф Степанович Есаян; Институт развития дошкольного образования. – М., 2005. – 168 с.

Zahryvyi, V. I. (1999). *Developing Senior Students' Cognitive Interest in Economic Knowledge in Extracurricular Activities.* (Abstract of the PhD dissertation). Pedagogy Institute of Ukraine APS, Kyiv, Ukraine. (In Ukr.).
Загривий В. І. Формування у старшокласників пізнавального інтересу до економічних знань у позакласній роботі : автореф. дис. … канд. пед. наук : спец. 13.00.01 «Загальна педагогіка та історія педагогіки» / Володимир Іванович Загривий ; Ін-т педагогіки АПН України. – К., 1999. – 18 с.

Zaytseva, L. I. (2005). *Developing the elementary mathematical competence of the children of the senior preschool age.* (Unpublishing Doctoral dissertation). NAPS Institute of Problems of Education of Ukraine, Kiev, Ukraine. (In Ukr.)
Зайцева Л. І. Формування елементарної математичної компетентності в дітей старшого дошкільного віку : дис. … канд. пед. наук: спец. 13.00.08 / Л. І.Зайцева. – Київ, Інститут проблем виховання НАПН України, 2005. – 203 с.

Zhovtan, L. (2000). *The differentiation in training the 7-11th grades in the process of in-depth study of subjects of natural-mathematical cycle.* (Abstract of the PhD dissertation). Lugansk state pedagogical university named after Taras Shevchenko, Lugansk, Ukraine. (In Ukr.).
Жовтан Л. В. Диференціація навчання учнів 7-11 класів у процесі поглибленого вивчення предметів природничо-математичного циклу: автореф. дис. на здобуття наук. ступеня канд. пед. наук: 13.00.09 / Л. В. Жовтан. – Х., 2001. – 22 с.

Zhytieniova, N.V. (2009). *Developing Cognitive Interest of the 7-9th Form Students in Teaching Natural Science and Mathematics with Computer Support.* (Abstract of the PHD Dissertation) G. S. Skovoroda National Pedagogical University, Kharkiv, Ukraine. (In Ukr.).
Житєньова Н. В. Формування пізнавального інтересу учнів 7–9 класу у процесі навчання предметів природничо-математичного циклу за комп'ютерної підтримки : дис. … канд. пед. наук : спец. 13.00.09 «Теорія навчання» / Н. В. Житєньова; Харк. нац. пед. ун-т ім. Г. С. Сковороди. – Харків, 2009. – 265 с.

Zimniaya, I. (1997). *Pedagogical psychology*. Rostov-na-Donu, Russia: Phoenix. (In Rus.).
Зимняя И. А. Педагогическая психология : учеб. пособ. [для студ. учеб. завед.] / Ирина Алексеевна Зимняя. – Ростов-на-Дону : Феникс, 1997. – 480 с.

Znakov, V. V. (2009). Three traditions of psychological research – three types of understanding. *Questions of psychology, 4*, 14–23 (In Rus.).
Знаков В. В. Три традиции психологических исследований – три типа понимания / В. В. Знаков // Вопросы психологии. – 2009. – № 4. – С. 14–23.

Zuckerman, V. (2012). Kolmogorov's Reforma and school mathematics education today. *Library "Moscow State University for School"*. Retrieved from http://lib.teacher.msu.ru/pub/2330 (In Rus.).
Цукерман В. В. Реформа А. Н. Колмогорова и школьное математическое образование сегодня / В. В. Цукерман // Библиотека "МГУ-школе" : [Электронный ресурс]. Режим доступа: http://lib.teacher.msu.ru/pub/2330

OUR AUTHORS

Akulenko Irina – Doctor (habil.) of Science in Pedagogy, Full Professor,
Department of Algebra and Mathematics Analysis,
Bohdan Khmelnytsky National University of Cherkasy, *Ukraine*
e-mail: akulenkoira@mail.ru

Bochko Oksana – PhD in Pedagogy and Education, associate Professor,
Department of mathematics and methods of teaching mathematics,
Bohdan Khmelnytsky National University of Cherkasy, *Ukraine*
e-mail: ok.volovik@gmail.com

Bogatyreva Iryna – PhD, in Pedagogy and Education, associate Professor,
Department of mathematics and methods of teaching mathematics,
Bohdan Khmelnytsky National University of Cherkasy, *Ukraine*
e-mail: i_bogatyreva@ukr.net

Bosovskiy Mykola – PhD in Pedagogy and Education, associate Professor,
Department of mathematics and methods of teaching mathematics,
Bohdan Khmelnytsky National University of Cherkasy, *Ukraine*
e-mail: bosovskyy@gmail.com

Bozhko Alina – PhD student in Pedagogy and Education,
Department of mathematics and methods of teaching mathematics,
Bohdan Khmelnytsky National University of Cherkasy, *Ukraine*
e-mail: bozhko.alina@inbox.ru

Brezhneva Helen – PhD in Pedagogy and Education, associate Professor,
Department of Preschool Education of Mariupol State University,
doctoral student of Institute of Problems of Education of National Academy of Pedagogical Sciences of Ukraine, *Ukraine*
e-mail: elena_brezhneva@mail.ru

Cernajeva Sarmite – Mg. math., Mg. ped., lector,
Riga Technical University, *Latvia*
e-mail: sarmite.cernajeva@inbox.lv

Donets Mykhailo – PhD student in Pedagogy and Education,
Department of mathematics and methods of teaching mathematics,
Bohdan Khmelnytsky National University of Cherkasy, *Ukraine*
e-mail: donets_misha@ukr.net

Golodiuk Larysa – PhD in Pedagogy and Education, associate Professor,
deputy director for scientific and methodological activities,
Municipal Institution "Kirovograd Regional IN-Service Teacher Training Institute named after Vasyl Sukhomlynsky", *Ukraine*
e-mail: laragolodiuk@gmail.com

Kolomiets Oksana – PhD in Pedagogy and Education, associate Professor,
Department of mathematics and methods of teaching mathematics,
Bohdan Khmelnytsky National University of Cherkasy, *Ukraine*
e-mail: ok_kolomiec71@ukr.net

Lovyanova Iryna – Doctor (habil.) of Science in Pedagogy, Associate Professor,
Kryvyi Rih Pedagogical Institute State higher education institution "Kryvyi Rih National University", *Ukraine*
e-mail: lovira22@i.ua

Lutsenko Galyna – PhD, associate Professor,
doctoral student of Sciences in Pedagogy,
Department of mathematics and methods of teaching mathematics,
Bohdan Khmelnytsky National University of Cherkasy, *Ukraine*
e-mail: LutsenkoG@gmail.com

Milloushev Vassil – Doctor (habil.) of Science in Pedagogy, Full Professor,
Faculty of Mathematics and Informatics,
Plovdiv University "Paisii Hilendarski", *Bulgaria*
e-mail: milushev_vassil@abv.bg

Millousheva-Boikina Dobrinka – PhD in Pedagogy and Education, associate Professor,
Faculty of Mathematics and Informatics,
Plovdiv University "Paisii Hilendarski", *Bulgaria*
e-mail: boykina@uni-plovdiv.bg; boykin@abv.bg

Serdiuk Zoya – PhD in Pedagogy and Education, Associate Professor,
Department of mathematics and methods of teaching mathematics,
Bohdan Khmelnytsky National University of Cherkasy, *Ukraine*
e-mail: serdyuk_z@ukr.net

Tarasenkova Nina – Doctor (habil.) of Science in Pedagogy, Full Professor,
Department of mathematics and methods of teaching mathematics,
Bohdan Khmelnytsky National University of Cherkasy, *Ukraine*
e-mail: ntaras7@ukr.net

Terekh Oksana – PhD student in Pedagogy and Education,
Department of mathematics and methods of teaching mathematics,
Bohdan Khmelnytsky National University of Cherkasy, *Ukraine*
e-mail: oksanka_5-5-5_@mail.ru

Tereshchenko Viktoria – PhD student in Pedagogy and Education,
Department of mathematics and methods of teaching mathematics,
Bohdan Khmelnytsky National University of Cherkasy, *Ukraine*
e-mail: viktoria_m21@mail.ru

Tretyak Mykola – PhD in Pedagogy and Education, Associate Professor,
Department of mathematics and methods of teaching mathematics,
Bohdan Khmelnytsky National University of Cherkasy, *Ukraine*
e-mail: mykola.tretyak@gmail.com

Vasilenko Iryna – PhD in Pedagogy and Education, teacher of mathematics,
Cherkassy Medical College, *Ukraine*
e-mail: vasilenko.rrina@mail.ru

Volodko Inta – Dr. math., Full Professor,
Riga Technical University, *Latvia*
e-mail: inta.volodko@rtu.lv

Accessible at http://seanewdim.com/other-publications.html

Conceptual framework for improving the mathematical training of young people: Monograph

2016, 212 p., Hardcover, 500 copies
ISBN: 978-963-12-7666-4

Managing editors:
Barkáts, N.
& Diadyk, A.

Készült a Rózsadomb Contact Kft nyomdájában.
1022 Budapest, Balogvár u. 1.
www.rcontact.hu

BUDAPEST

2016

www.seanewdim.com