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"Predator and prey" model revisited – influence of external fluxes and noise

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(*Presented by I. I. Skrypnik*)

Abstract. The well-known predator-prey model is modified in two ways. First, the regular adding or regular deleting of preys or/and predators is considered. The steady-state and stability diagram are found. Second, random fluctuations of the birthrate and other kinetic coefficients are studied, and the parabolic law of a random walk in the (X,Y)-space is found and proved for small deviations from the steady state.

Keywords. Lotka–Volterra model, nonlinear differential equations, stability analysis, noise.

1. Introduction

The predator-prey model introduced by Lotka and Volterra is a basic synergetic model demonstrating the oscillatory behavior of nonlinear biological, chemical, or economic systems [1–4]. It is governed by the simple system of two nonlinear equations with regard for the natural birthrate $\bar{k}_1$ of preys, natural death rate $\bar{k}_4$ of predators, as well as "collisions" of preys with predators, unlucky for preys (rate $\bar{k}_2$) and lucky for feeding the new generations of predators (rate $\bar{k}_3$):

$$\begin{aligned} \frac{d\bar{X}}{d\bar{t}} &= \bar{k}_1\bar{X} - \bar{k}_2\bar{X}\bar{Y}, \\ \frac{d\bar{Y}}{d\bar{t}} &= \bar{k}_3\bar{X}\bar{Y} - \bar{k}_4\bar{Y} \end{aligned} \tag{1.1}$$

(X is the number of preys, and Y is the number of predators). If the "CREATOR" of this ecosystem, by choosing the initial numbers of both species,"misses" the stationary numbers $\bar{X}^{st} = \bar{k}_4/\bar{k}_3$ and $\bar{Y}^{st} = \bar{k}_1/\bar{k}_2$, the system demonstrates the oscillatory behavior, all oscillations proceeding around the mentioned stationary point. The transition to dimensionless variables, $X = \bar{X}/\bar{X}^{st}, Y = \bar{Y}/\bar{Y}^{st}, t = \sqrt{\bar{k}_1\bar{k}_4}\bar{t}$, gives the system of two universal equations:

$$\begin{aligned} \frac{1}{a}\frac{dX}{dt} &= k_1X - k_2X \cdot Y \\ a\frac{dY}{dt} &= k_3X \cdot Y - k_4Y \end{aligned} \tag{1.2}$$

with $a = \sqrt{\bar{k}_1/\bar{k}_4}, k_1 = 1, k_2 = 1, k_3 = 1, k_4 = 1$. This model is very idealized and almost closed (excluding the unlimited feed for preys). It is the main cause for why the phase trajectories in the

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standard Lotka–Volterra (LV) model neither converge nor diverge, but just oscillate. Mathematically, this means that the linearized system of equations for deviations,

$$\begin{aligned}\frac{d\delta X}{dt} &= a\cdot((1-Y^{st})\delta X - X^{st}\cdot\delta Y) = 0\cdot\delta X + (-a)\cdot\delta Y\\ \frac{dY}{dt} &= \frac{1}{a}(Y^{st}\delta X + (X^{st}-1)\cdot\delta Y) = \frac{1}{a}\cdot\delta X + 0\cdot\delta Y\end{aligned}\tag{1.3}$$

provides the purely imaginary (with zero real part) roots of the characteristic equation: $\det\left\|\begin{matrix}-\lambda & -a\\ 1/a & -\lambda\end{matrix}\right\| = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$, which means oscillations without divergence or convergence.

Everywhere below, we will limit ourselves to the particular, most symmetric case $a = 1$.

There exist a lot of modifications and generalizations of the predator-prey model [5–9], including the inhomogeneity of a system and the account for the diffusion, several types of preys or/and predators, noise (fluctuations) of the numbers of preys or/and predators. In this paper, we suggest two more ways of modifications, which seem natural:

1. We will introduce sponsors/hunters of predators or/and preys with a "license" on a constant rate of sponsoring/hunting (regular external fluxes).

2. We will introduce the noise of kinetic coefficients.

We will see that the introduction of adding or deleting predators or/and preys substantially broadens the spectrum of possible regimes: (I) the system may remain eternally oscillating without convergence or divergence, as in the classical LV-model, (II) system can be stable and converge to the steady-state limit, (III) system can be metastable, by converging to a steady state from the initial positions in some critical vicinity of the stationary solution and diverging from the positions outside this critical region, and (IV) system can be totally unstable and always diverging.

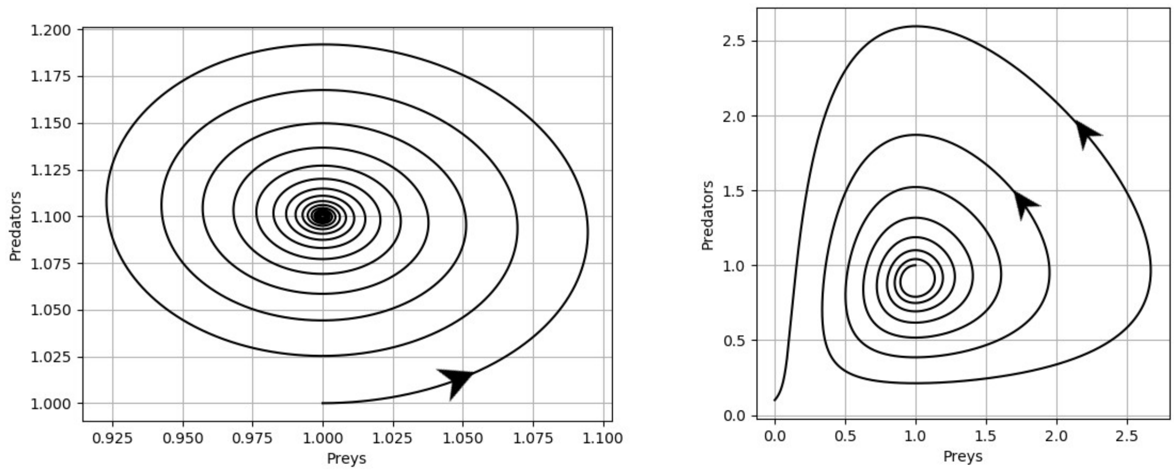
We will also see that the noise of kinetic coefficients, on the average, leads to the divergence, but the time law for the growing mean squared distance from the stationary solution is peculiar and resembles the Brownian motion.

2. Influence of regular adding/hunting

The basic equations for LV-system with external fluxes are:

$$\begin{aligned}\tfrac{dX}{dt} &= X - X\cdot Y + b_x,\\ \tfrac{dY}{dt} &= X\cdot Y - Y + b_y.\end{aligned}\tag{2.1}$$

We will start from some simple examples. The regular adding of preys without predators being touched ($b_x > 0, b_y = 0$) stabilizes the system (Fig.1a), whereas the regular hunting of preys without predators being touched ($b_x < 0, b_y = 0$) destabilizes the system (Fig. 1b):

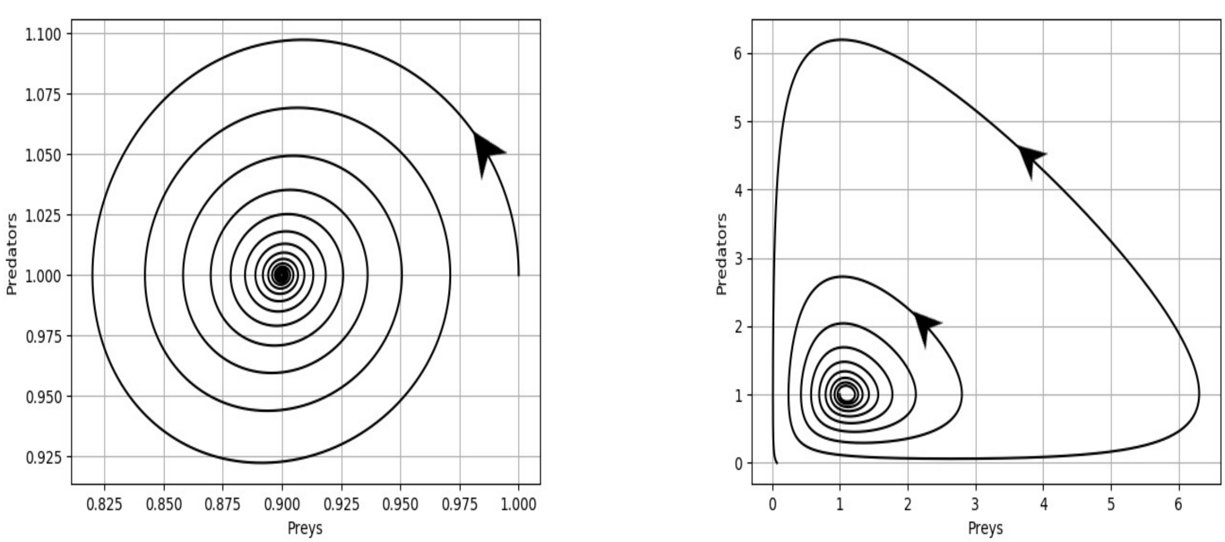


(a) $b_x = 0.1, b_y = 0$

(b) $b_x = -0.1, b_y = 0$

Figure 1: Phase trajectories in the cases of the adding and hunting of preys.

In the cases of the adding/hunting of only predators without preys being touched, we have analogous situation (Fig. 2), but the adding of predators stabilizes the system only for $0 < b_y < 1$.



(a) $b_x = 0, b_y = 0.1$

(b) $b_x = 0, b_y = -0.1$

Figure 2: Phase trajectories in the cases of the adding and hunting of predators.

Now, we will consider the general case. The equations for stationary points have the formal solution

$$X^{st} = \frac{1-b_x-b_y}{2} + \sqrt{\frac{(1-b_x-b_y)^2}{4} + b_x}, Y^{st} = \frac{1+b_x+b_y}{2} + \sqrt{\frac{(1-b_x-b_y)^2}{4} + b_x}. \quad (2.2)$$

Of course, only the positive solutions ($X^{st} > 0, Y^{st} > 0$) should be further considered. Moreover, these solutions should be at least locally stable. To find the criteria of local stability, we linearize Eq. (2.1) in a vicinity of the stationary solution determined by Eq. (2.2):

$$\frac{d\delta X}{dt} = (1-Y^{st})\delta X - X^{st}, \cdot \delta Y$$
$$\frac{dY}{dt} = Y^{st}\delta X + (X^{st}-1)\cdot \delta Y. \quad (2.3)$$

Then the local stability is determined by the real parts of roots of the characteristic equation

$$\det \begin{Vmatrix} 1-Y^{st}-\lambda & -X^{st} \\ Y^{st} & X^{st}-1-\lambda \end{Vmatrix} = 0 \Rightarrow \lambda^2 - (X^{st}-Y^{st})\lambda + X^{st} + Y^{st} - 1 = 0,$$

$$\lambda_1 = \frac{1}{2}((X^{st}-Y^{st}) + \sqrt{(X^{st}-Y^{st})^2 + 4(1-X^{st}-Y^{st})})$$
$$\lambda_2 = \frac{1}{2}((X^{st}-Y^{st}) - \sqrt{(X^{st}-Y^{st})^2 + 4(1-X^{st}-Y^{st})}). \quad (2.4)$$

The result of steady-state analysis is summarized in Fig. 3.

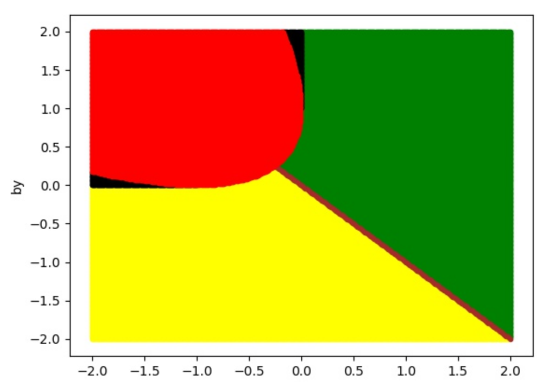
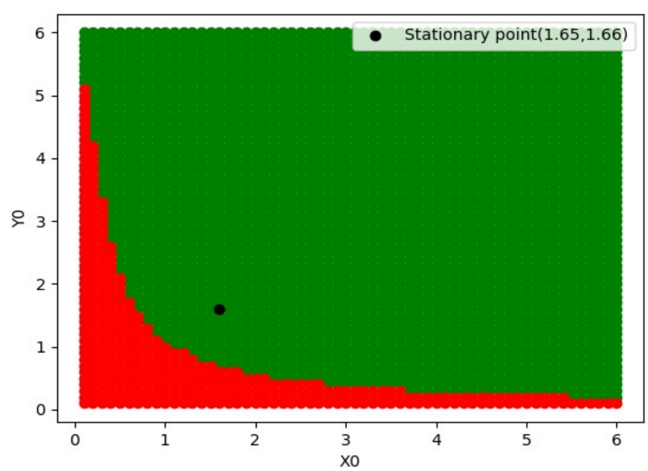


Figure 3: Diagram (b_x, b_y) of regimes in the open Lotka–Volterra system.

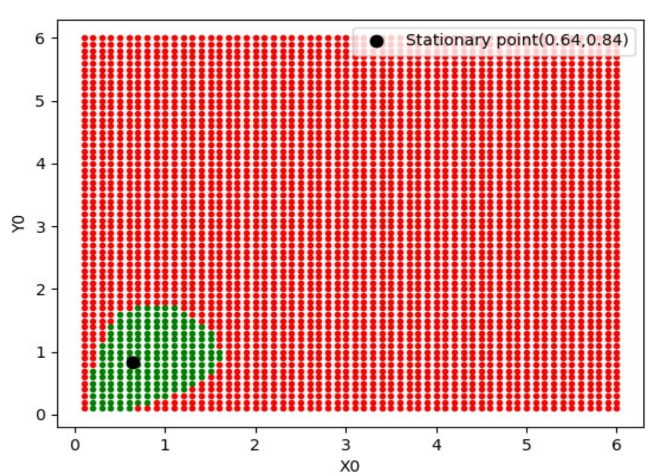
Red – negative discriminant, $\frac{(1-b_x-b_y)^2}{4} + b_x < 0$ – no stationary solutions at all.
Black – negative or zero values of X^{st} or Y^{st} – nonphysical stationary solutions.
Yellow – physical stationary solutions, but unstable (at least one of the real parts of λ is positive or zero).
Brown – oscillatory behavior without convergence or divergence (zero stability).
Green – locally stable stationary solutions.

In general, we should distinguish globally stable and locally stable (metastable) stationary solutions. This will be done elsewhere.

Two examples are given in Fig. 4(a, b). Namely, we demonstrate the convergence region (green) in the X-Y plane around the metastable stationary points corresponding to the following cases: (a) $b_x = 1.1, b_y = -1.09, X^{st} = 1.65, Y^{st} = 1.66$, (b) $b_x = -0.1, b_y = 0.3, X^{st} = 0.64, Y^{st} = 0.84$.



(a) $b_x = 1.1, b_y = -1.09, X^{st} = 1.65, Y^{st} = 1.66$



(b) $b_x = -0.1, b_y = 0.3, X^{st} = 0.64, Y^{st} = 0.84$

Figure 4: Stability/collapse diagram (stationary point is marked with black spot, convergence region is green)

3. Noise of kinetic coefficients

The influence of a noise on the behavior of the LV system is not a new problem. As far as we know, only the noises of X and Y had been explored, namely, the random adding or deleting of a small number of preys or predators [5, 8]. Additionally to this, we will explore random fluctuations of the kinetic coefficients k_1, k_2, k_3, and k_4 (for example, fluctuations of the birthrate due to rainy days).

We start from the Langevin noise of a reduced birthrate without memory and with fixed amplitude A:

$$\begin{aligned} &k_1 = 1 + \xi(t) \\ &< \xi(t)\xi(t') >= A^2\delta(t - t'). \end{aligned} \tag{3.1}$$

In the case of numerical modeling, one should introduce this noise in such a way that a change in the time step should not change the noise impact. Our suggestion is a stepwise probability distribution:

$$k_1 = 1 + \frac{A}{\sqrt{dt}}\sqrt{3}(2 \cdot random - 1), \tag{3.2a}$$

(The mean square value of the random function $\sqrt{3}(2 \cdot random - 1)$ is equal to 1). Alternatively, one may use the Gaussian distribution

$$k_1 = 1 + \frac{A}{\sqrt{dt}} \sin(2\pi\, random)\sqrt{2\ln(1/random)}. \tag{3.2b}$$

In more details, the introduction of the noise of kinetic coefficients is discussed in [9, 10] in the case of atomic migration. We start from the stationary point $(X = 1, Y = 1)$ as an initial condition. The typical phase trajectory as a numerical solution of the system

$$\begin{aligned} &\frac{dX}{dt} = (1 + \xi(t))X - X \cdot Y \\ &\frac{dY}{dt} = X \cdot Y - Y. \end{aligned} \tag{3.3}$$

is shown in Fig. 5.

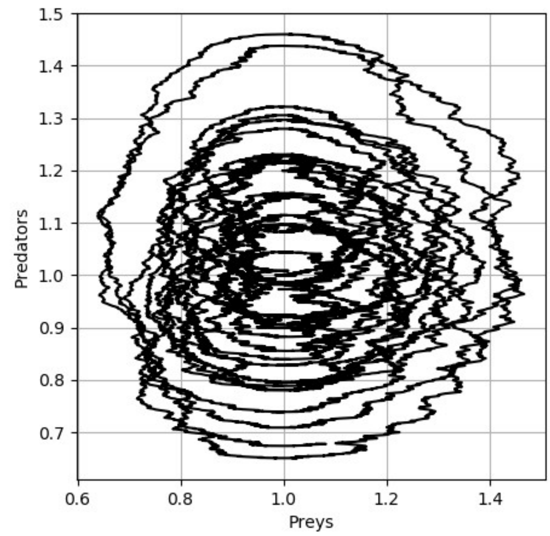


Figure 5: Typical phase trajectory of the LV-system with a fluctuating birthrate; it starts from the stationary point.

Then we took the ensemble of $M = 100$ LV-systems originating at the same stationary point (1,1), and found the mean square displacement from this point as a function of the time. The results for different time steps and the same amplitude are shown in Fig. 6.

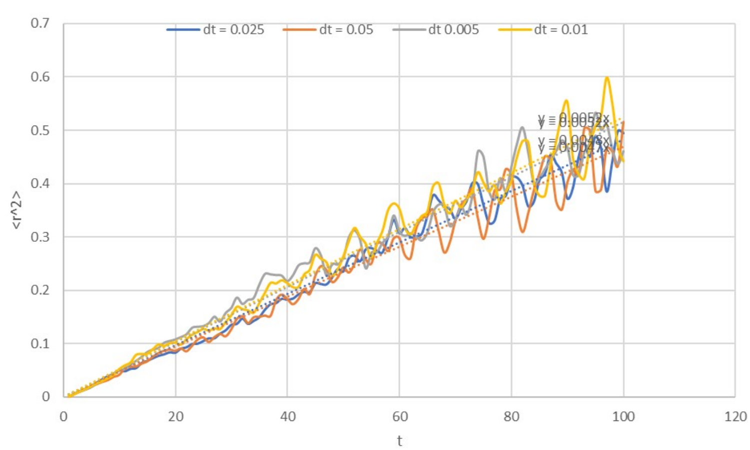


Figure 6: Noise of the birthrate k_1. Mean square displacement from the stationary point versus the time for the same amplitude A=0.07 and various time steps dt. In all cases, the trajectory starts from the stationary points

One can see that the dependences for different time steps are close to one another and can be approximated as

$$< (\Delta X)^2 + (\Delta Y)^2 >\approx \alpha A^2 t, \alpha \approx 1. \tag{3.4}$$

If the initial point differs from the stationary point, then the initial mean square displacement firstly decreases and then follows the same time law – see Fig. 7.

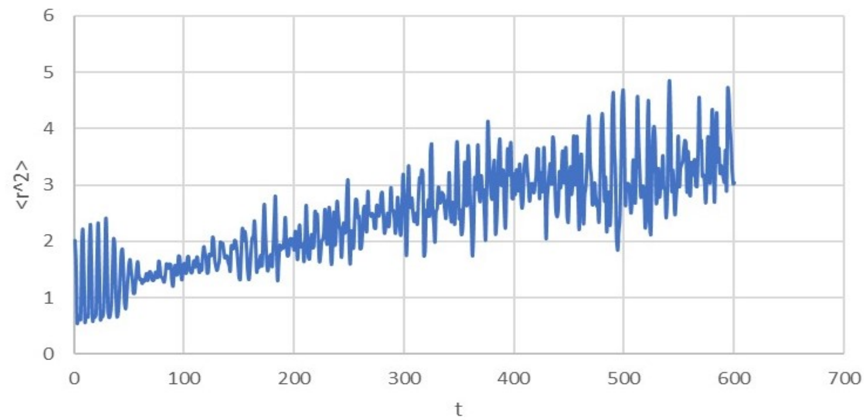


Figure 7: Noise of the birthrate k_1. Mean square displacement from the stationary point versus the time with an initial deviation from the stationary point ($A = 0.07, dt = 0.05, X_0 = 2, Y_0 = 2$).

Analogous dependences were obtained by numerical simulations for the noise of k_2, k_3, and k_4.

Most probably, the parabolic dependence is related to the "zero stability" of the LV system. So, we observe something like a random walk in the XY space.

We can solve Eq. (3.4) analytically and find the exact value of α at least for small deviations from the steady state, by using linearized kinetic equations.

Theorem 1. *The linearization (first-order approximation) of the Lotka–Volterra model in a vicinity of the steady state with the Langevin noise of the birthrate of preys and without external fluxes,*

$$\frac{dX}{dt} = (1 + \xi(t))X - X \cdot Y, \frac{dY}{dt} = X \cdot Y - Y,$$
$$< \xi(t)\xi(t') >= A^2\delta(t - t'), \tag{3.5}$$

gives the following parabolic law for the sum of dispersions for the ensemble of LV-systems:

$$< (\delta X)^2 > + < (\delta Y)^2 >= 1 \cdot A^2 t, \tag{3.6}$$

under the condition $t << 1/A^2$.

Proof. Consider the first-order approximation for deviations from the steady state: $X = 1 + \delta X, Y = 1 + \delta Y, |\delta X| << 1, |\delta Y| << 1$, so that $|\delta X \delta Y| << |\delta X|\,,\ |\delta X \delta Y| << |\delta Y|$. We have

$$\frac{d\delta X}{dt} = \xi(t) - \delta Y, \tag{3.7a}$$
$$\frac{d\delta Y}{dt} = \delta X. \tag{3.7b}$$

We consider an ensemble of LV-systems with identical initial condition

$$\delta X(t = 0) = 0, \delta Y(t = 0) = 0.$$

Let us multiply Eq. (3.7a) by $2\delta X$ and average over the ensemble:

$$\frac{d < (\delta X)^2 >}{dt} = -2 < \delta X \delta Y > +2 < \delta X(t)\xi(t) > . \tag{3.8a}$$

We now multiply Eq. (3.7b) by $2\delta Y$ and average over the ensemble:

$$\frac{d < (\delta Y)^2 >}{dt} = 2 < \delta X \delta Y > . \tag{3.8b}$$

Now, we add Eqs. (3.8a) and (3.8b) and get

$$\frac{d}{dt}(< (\delta X)^2 > + < (\delta Y)^2 >) = 2 < \delta X(t)\xi(t) > . \tag{3.9}$$

To make system (3.8a), (3.8b) self-consistent, one should find the value of $< \delta X(t)\xi(t) >$.

For this, first of all, we find the formal solution of system (3.7a), (3.7b) which is reformulated in the matrix form:

$$\frac{d}{dt}\begin{pmatrix}\delta X\\ \delta Y\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\delta X\\ \delta Y\end{pmatrix} + \begin{pmatrix}\xi\\ 0\end{pmatrix} \equiv \hat{M}\begin{pmatrix}\delta X\\ \delta Y\end{pmatrix} + \hat{f},$$
$$\delta X(t=0) = 0, \delta Y(t=0) = 0. \tag{3.10}$$

A solution of Eq. (3.10) is as follows:

$$\begin{pmatrix}\delta X(t)\\ \delta Y(t)\end{pmatrix} = \int_0^t \exp((t-t')\hat{M})\begin{pmatrix}\xi(t')\\ 0\end{pmatrix} dt'. \tag{3.11}$$

Then we can find two mean values $< \delta X(t)\xi(t) >$ and $< \delta Y(t)\xi(t) >$:

$$< \begin{pmatrix}\delta X(t)\\ \delta Y(t)\end{pmatrix}\xi(t) >= \begin{pmatrix}< \delta X(t)\xi(t) >\\ < \delta Y(t)\xi(t) >\end{pmatrix} = \int_0^t \exp((t-t')\hat{M})\cdot$$
$$\cdot < \begin{pmatrix}\xi(t')\\ 0\end{pmatrix}\xi(t) > dt' = \int_0^t \exp((t-t')\hat{M})\begin{pmatrix}A^2\delta(t-t')\\ 0\end{pmatrix} dt' =$$
$$= \begin{pmatrix}A^2/2\\ 0\end{pmatrix}. \tag{3.12}$$

(Note that the factor 1/2 appeared, because the argument of the delta function $\delta(t-t')$ is equal to zero not within the integration interval, but instead at its boundary.) Thus,

$$< \xi(t)\delta X(t) >= A^2/2,$$
$$\frac{d}{dt}(< (\delta X)^2 > + < (\delta Y)^2 >) = A^2 \Rightarrow < (\delta X)^2 > + < (\delta Y)^2 >= A^2 t. \tag{3.13}$$

We recall that all this derivation is valid only for linearized predator-prey equations. The linearization is valid only under the conditions $|\delta X| << 1,\ |\delta Y| << 1 \Leftrightarrow (\delta X)^2 + (\delta Y)^2 << 1$.

For this, the time should be short enough: $t << 1/A^2$.

Theorem 1 is proved. □

Now, we consider the noise of the birthrate of predators.

Theorem 2. *The linearization of the Lotka–Volterra model with the Langevin noise of the birthrate of predators and without external fluxes,*

$$\frac{dX}{dt} = X - X\cdot Y, \frac{dY}{dt} = X\cdot Y - (1+\xi(t))Y, < \xi(t)\xi(t') >= A^2\delta(t-t'), \tag{3.14}$$

leads to the parabolic law (3.6) *for the sum of dispersions.*

Proof. Again, we consider the first-order approximation for deviations from the steady state: $X = 1 + \delta X, Y = 1 + \delta Y, |\delta X| << 1, |\delta Y| << 1$,

$$\frac{d\delta X}{dt} = -\delta Y \tag{3.15a}$$

$$\frac{d\delta Y}{dt} = \delta X - \xi(t). \tag{3.15b}$$

We consider an ensemble of LV-systems with identical initial condition

$$\delta X(t = 0) = 0, \delta Y(t = 0) = 0.$$

Let us multiply Eq. (3.15a) by $2\delta X$ and average over the ensemble:

$$\frac{d < (\delta X)^2 >}{dt} = -2 < \delta X \delta Y > . \tag{3.16a}$$

Let us multiply Eq. (3.15b) by $2\delta Y$ and average over the ensemble:

$$\frac{d < (\delta Y)^2 >}{dt} = 2 < \delta X \delta Y > -2 < \delta Y \cdot \xi > . \tag{3.16b}$$

Now, we add Eqs. (3.16a) and (3.16b) and obtain

$$\frac{d}{dt}(< (\delta X)^2 > + < (\delta Y)^2 >) = -2 < \delta Y(t)\xi(t) > . \tag{3.17}$$

Now, one should find the value of $< \delta Y(t)\xi(t) >$.

For this, first of all, we find the formal solution of system (3.15a), (3.15b) which we reformulate in the matrix form:

$$\frac{d}{dt}\begin{pmatrix}\delta X\\ \delta Y\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\delta X\\ \delta Y\end{pmatrix} + \begin{pmatrix}0\\ -\xi\end{pmatrix} \equiv \hat{M}\begin{pmatrix}\delta X\\ \delta Y\end{pmatrix} + \hat{f},$$
$$\delta X(t = 0) = 0, \delta Y(t = 0) = 0. \tag{3.18}$$

A solution of Eq. (3.18) reads

$$\begin{pmatrix}\delta X(t)\\ \delta Y(t)\end{pmatrix} = \int_0^t \exp((t - t')\hat{M})\begin{pmatrix}0\\ -\xi(t')\end{pmatrix} dt'. \tag{3.19}$$

Then we can find two mean values $< \delta X(t)\xi(t) >$ and $< \delta Y(t)\xi(t) >$:

$$\begin{aligned}
&< \begin{pmatrix}\delta X(t)\\ \delta Y(t)\end{pmatrix} \xi(t) >= \begin{pmatrix}< \delta X(t)\xi(t) >\\ < \delta Y(t)\xi(t) >\end{pmatrix} =\\
&= \int_0^t \exp\left((t - t')\hat{M}\right) < \begin{pmatrix}0\\ -\xi(t')\end{pmatrix} \xi(t) > dt' =\\
&= \int_0^t \exp\left((t - t')\hat{M}\right) \begin{pmatrix}0\\ -A^2\delta(t - t')\end{pmatrix} dt' = \begin{pmatrix}0\\ -A^2/2\end{pmatrix}.
\end{aligned} \tag{3.20}$$

Thus,

$$< \delta Y(t)\xi(t) >= -A^2/2 \tag{3.21}$$

$$\frac{d}{dt}\left(< (\delta X)^2 > +< (\delta Y)^2 >\right) = A^2 \Rightarrow < (\delta X)^2 > +< (\delta Y)^2 >= A^2 t.$$

The theorem is proved. □

Now, we consider the noise of the extinction rate of preys (caught by predators).

Theorem 3. *The linearization of the Lotka–Volterra model with the Langevin noise of the extinction rate of preys and without external fluxes gives*

$$\frac{dX}{dt} = X - (1 + \xi(t))\, X \cdot Y,\ \frac{dY}{dt} = X \cdot Y - Y,$$
$$< \xi(t)\, \xi(t') >= A^2 \delta(t - t') \tag{3.22}$$

and the parabolic law (3.6) *for the sum of dispersions.*

Proof. Again, consider the first-order approximation for deviations from the steady state: $X = 1 + \delta X,\ Y = 1 + \delta Y,\ |\delta X| << 1,\ |\delta Y| << 1,$

$$\frac{d\delta X}{dt} = -\xi(t) - \delta Y, \tag{3.23a}$$
$$\frac{d\delta Y}{dt} = \delta X. \tag{3.23b}$$

Equations (3.23a) and (3.23b) are identical with Eqs. (3.7a) and (3.7b) except for the sign before the noise term. All characteristics (probabilities and mean values) of the Langevin noise are symmetric in the sign. Therefore, Theorem 3 directly follows from Theorem 1. □

Now, we pass to the consideration of the noise of a reproduction rate of predators due to the eating of preys.

Theorem 4. *The linearization of the Lotka–Volterra model with the Langevin noise of a reproduction rate of predators due to the eating of preys and without external fluxes,*

$$\frac{dX}{dt} = X - X \cdot Y,\ \frac{dY}{dt} = (1 + \xi(t))\, X \cdot Y - Y,$$
$$< \xi(t)\, \xi(t') >= A^2 \delta(t - t'), \tag{3.24}$$

gives the parabolic law (3.6) *for the sum of dispersions.*

Proof. Again, we consider the first-order approximation for deviations from the steady state: $X = 1 + \delta X,\ Y = 1 + \delta Y,\ |\delta X| << 1,\ |\delta Y| << 1$,

$$\frac{d\delta X}{dt} = -\delta Y, \tag{3.25a}$$
$$\frac{d\delta Y}{dt} = \delta X + \xi(t). \tag{3.25b}$$

Equations (3.25a) and (3.25b) are identical with Eqs. (3.15a) and (3.15b) except for the sign before the noise term. All characteristics (probabilities and mean values) of the Langevin noise are symmetric in respect to the sign. Therefore, Theorem 4 directly follows from Theorem 2. □

Now, we proceed to the noise in the LV-model with external fluxes, by adding a noise to various terms in Eqs. (2.1). We start with the simplest cases where the external flux is present, but its mean value is zero.

Theorem 5. *The linearization of the Lotka–Volterra model with the Langevin noise of the external flux of preys,*

$$\frac{dX}{dt} = X - X \cdot Y + \xi(t), \; \frac{dY}{dt} = X \cdot Y - Y,$$
$$< \xi(t)\xi(t') >= A^2\delta(t - t') \tag{3.26}$$

leads to the parabolic law (3.6) *for the sum of dispersions.*

Proof. Again, consider the first-order approximation for deviations from the steady state: $X = 1 + \delta X, \; Y = 1 + \delta Y, \; |\delta X| << 1, \; |\delta Y| << 1$,

$$\frac{d\delta X}{dt} = -\delta Y + \xi(t), \tag{3.27a}$$
$$\frac{d\delta Y}{dt} = \delta X. \tag{3.27b}$$

Equations (3.27a) and (3.27b) are identical with (3.7a) and (3.7b). Therefore, Theorem 5 directly follows from Theorem 1. □

Theorem 6. *The linearization of the Lotka–Volterra model with the Langevin noise of the external flux of predators,*

$$\frac{dX}{dt} = X - X \cdot Y, \; \frac{dY}{dt} = X \cdot Y - Y + \xi(t),$$
$$< \xi(t)\xi(t') >= A^2\delta(t - t'), \tag{3.28}$$

gives the parabolic law (3.6) *for the sum of dispersions.*

Proof. Again, consider the first-order approximation for deviations from the steady state: $X = 1 + \delta X, \; Y = 1 + \delta Y, \; |\delta X| << 1, \; |\delta Y| << 1$

$$\frac{d\delta X}{dt} = -\delta Y, \tag{3.29a}$$
$$\frac{d\delta Y}{dt} = \delta X + \xi(t). \tag{3.29b}$$

Equations (3.29a) and (3.29b) are identical with (3.25a) and (3.25b). Therefore, Theorem 6 directly follows from Theorem 4. □

Now, we proceed to the noise in LV-systems with nonzero mean external fluxes. Of course, the steady-state reduced values are not equal to 1 anymore in this case, but are determined by Eq. (2.2). In this paper, we consider only one case – the Lotka–Volterra model with the Langevin noise of the birthrate:

$$\frac{dX}{dt} = (1 + \xi(t))X - X \cdot Y + b_x, \; \frac{dY}{dt} = X \cdot Y - Y + b_y,$$
$$< \xi(t)\xi(t') >= A^2\delta(t - t'). \tag{3.30}$$

The linearization for first-order deviations from the steady state ($X = X^{st} + \delta X, \; Y = Y^{st} + \delta Y, \; |\delta X| << X^{st}, \; |\delta Y| << Y^{st}$) gives

$$\frac{d\delta X}{dt} = \xi(t)X^{st} + (1 - Y^{st})\delta X - X^{st}\delta Y, \tag{3.31a}$$
$$\frac{d\delta Y}{dt} = Y^{st}\delta X + (X^{st} - 1)\delta Y. \tag{3.31b}$$

We again consider an ensemble with identical initial condition $\delta X\,(t{=}0) = 0$, $\delta Y\,(t=0) = 0$. Let us multiply Eq. (3.31a) by $2\delta X$ and average over the ensemble:

$$\frac{d < (\delta X)^2 >}{dt} = +2\left(1 - Y^{st}\right) < (\delta X)^2 > -2X^{st} < \delta X \delta Y > \\ +2X^{st} < \delta X\,(t)\,\xi\,(t) > . \tag{3.32a}$$

Let us multiply Eq. (3.31b) by $2\delta Y$ and average over the ensemble:

$$\frac{d < (\delta Y)^2 >}{dt} = 2\left(X^{st} - 1\right) < (\delta Y)^2 > +2Y^{st} < \delta X \delta Y > . \tag{3.32b}$$

Let us add the product of Eq. (3.31a) with $2\delta Y$ and the product of Eq. (3.31b) with $2\delta X$ and average over the ensemble:

$$\frac{d < \delta X \delta Y >}{dt} = Y^{st} < (\delta X)^2 > -X^{st} < (\delta Y)^2 > + \\ + \left(X^{st} + Y^{st} - 2\right) < \delta X \delta Y > + < \delta Y\,(t)\,\xi\,(t) > . \tag{3.32c}$$

To make system (3.32a), (3.32b), (3.32c) self-consistent, one should find the values of $< \delta X\,(t)\,\xi\,(t) >$, $< \delta Y\,(t)\,\xi\,(t) >$.

For this, first of all, we find the formal solution of system (3.7a), (3.7b) which we reformulate in the matrix form:

$$\frac{d}{dt}\begin{pmatrix} \delta X \\ \delta Y \end{pmatrix} = \begin{pmatrix} \left(1 - Y^{st}\right) & -X^{st} \\ Y^{st} & \left(X^{st} - 1\right) \end{pmatrix}\begin{pmatrix} \delta X \\ \delta Y \end{pmatrix} + \begin{pmatrix} \xi \\ 0 \end{pmatrix} \equiv \\ \equiv \hat{M}\begin{pmatrix} \delta X \\ \delta Y \end{pmatrix} + \hat{f}, \\ \delta X\,(t=0) = 0,\ \delta Y\,(t=0) = 0.$$

The solution is:

$$\begin{pmatrix} \delta X\,(t) \\ \delta Y\,(t) \end{pmatrix} = \int\limits_0^t \exp\left(\left(t - t'\right)\hat{M}\right)\begin{pmatrix} \xi\,(t') \\ 0 \end{pmatrix} dt'.$$

Then we can find two mean values $< \delta X\,(t)\,\xi\,(t) >$ and $< \delta Y\,(t)\,\xi\,(t) >$. We have

$$< \begin{pmatrix} \delta X\,(t) \\ \delta Y\,(t) \end{pmatrix} \xi\,(t) >= \begin{pmatrix} < \delta X\,(t)\,\xi\,(t) > \\ < \delta Y\,(t)\,\xi\,(t) > \end{pmatrix} = \\ = \int\limits_0^t \exp\left(\left(t - t'\right)\hat{M}\right) < \begin{pmatrix} \xi\,(t') \\ 0 \end{pmatrix} \xi\,(t) > dt' = \\ = \int\limits_0^t \exp\left(\left(t - t'\right)\hat{M}\right)\begin{pmatrix} A^2\delta\,(t - t') \\ 0 \end{pmatrix} dt' = \begin{pmatrix} A^2/2 \\ 0 \end{pmatrix}. \tag{3.33}$$

Thus, the system of equations (3.32a), (3.32b), (3.32c) becomes self-consistent:

$$\frac{d}{dt}\begin{pmatrix} <(\delta X)^2> \\ <(\delta Y)^2> \\ <(\delta X \delta Y)> \end{pmatrix} =$$
$$= \begin{pmatrix} 2\left(1-Y^{st}\right) & 0 & -2X^{st} \\ 0 & 2\left(X^{st}-1\right) & 2Y^{st} \\ Y^{st} & -X^{st} & X^{st}+Y^{st}-2 \end{pmatrix} \begin{pmatrix} <(\delta X)^2> \\ <(\delta Y)^2> \\ <(\delta X \delta Y)> \end{pmatrix} +$$
$$+ \begin{pmatrix} 2X^{st}A^2/2 \\ 0 \\ 0 \end{pmatrix}. \tag{3.34}$$

The formal solution (in the matrix form) of Eq. (3.34) is:

$$\hat{\psi}(t) = \int_0^t \exp\left(\left(t-t'\right)\hat{L}\right)\hat{\varphi}\left(t'\right) dt', \ \hat{\psi}(t=0) = 0 \tag{3.35}$$

where

$$\hat{\psi}(t) = \begin{pmatrix} <(\delta X)^2> \\ <(\delta Y)^2> \\ <(\delta X \delta Y)> \end{pmatrix},$$
$$\hat{L} = \begin{pmatrix} 2\left(1-Y^{st}\right) & 0 & -2X^{st} \\ 0 & 2\left(X^{st}-1\right) & 2Y^{st} \\ Y^{st} & -X^{st} & X^{st}+Y^{st}-2 \end{pmatrix},$$
$$\hat{\varphi} = \begin{pmatrix} X^{st}A^2 \\ 0 \\ 0 \end{pmatrix}. \tag{3.36}$$

If the steady-state solution satisfies the local stability criterion ($\mathrm{Re}\lambda_1 < 0$, $\mathrm{Re}\lambda_2 < 0$ in Eqs. (2.1)), then system (3.34) should describe the competition between the dynamical tendency of attraction to the metastable steady state and the stochastic tendency of migration from this steady state. It is physically evident that, sooner or later, the stochastic (noise) will bring the system beyond the limits of metastability (beyond the convergence region). It is also evident that the "Mean Time To Failure" (MTTF) of the LV-system should depend on the noise amplitude.

Here, we demonstrate two numerical examples. In Fig. 8, one can see three time dependences of the mean square displacement from the stable stationary point for three different amplitudes for the noise of the birthrate of preys (Fig. 8)

In Fig. 9, we see the noise-initiated displacement from the unstable stationary point.

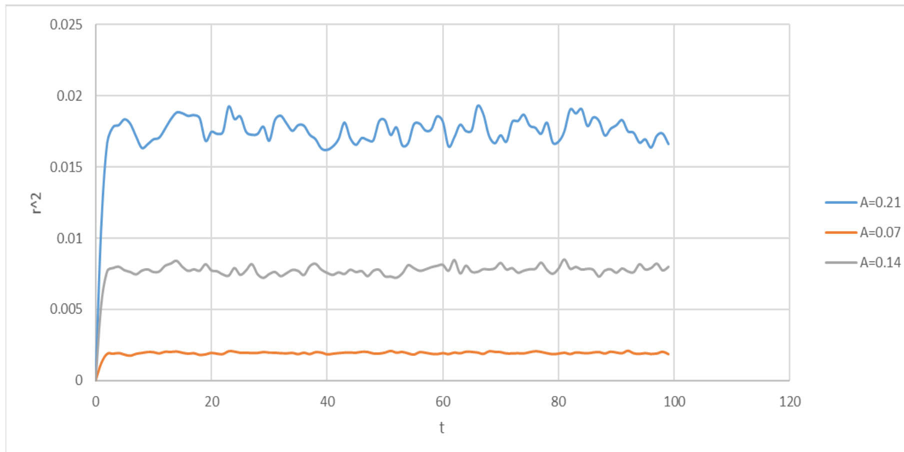


Figure 8: Time dependence of the mean square displacement from the stationary point corresponding to $b_x = b_y = 1$. Asymptotic levels are proportional to the squared noise amplitude of the birthrate of preys.

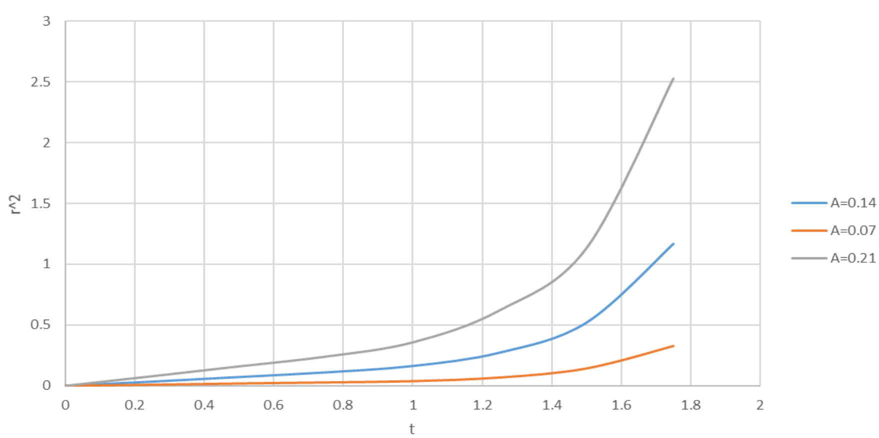


Figure 9: Time dependence of the mean square displacement from the stationary point corresponding to $b_x = b_y = -1$. Plots are calculated for three amplitudes of the noise of a birthrate.

We also observe the steady plateau with abrupt growth after some time. This problem will be considered elsewhere. (See also [11].)

Conclusion

1. Regular hunting of preys without troubling predators destabilizes the LV-system.
2. Regular hunting of predators without troubling preys destabilizes the LV-system.
3. Regular adding of preys always stabilizes the LV-system.
4. Regular adding of predators stabilizes the LV-system, only if the adding rate is less than some threshold.
5. Diagram of regimes for the LV-system with regular external fluxes is described by Fig. 3.

6. Simultaneous adding of preys and a limited hunting of predators may leave the system metastable, but only within some critical region of initial parameters.
7. In numerical modeling, to make the impact of the noise of kinetic parameters independent of the choice of a time step, the random perturbation of the kinetic coefficients should be proportional to the noise amplitude and inversely proportional to the square root of the time step.
8. Mean square distance from the stationary point (at least in the case of small deviations from the steady state without external fluxes) increases proportionally to the time with the proportionality coefficient proportional to the noise amplitude:$< (\Delta X)^2 + (\Delta Y)^2 >= A^2 t$
9. Competition between the noise and the metastability under nonzero external fluxes (including MTTF) will be analyzed elsewhere.

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